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REMARKS ON THE COSMOLOGICAL MODEL OF MILNE

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Abstract: The position often communicated in the literature that the curvature properties of an expanding cosmological model might depend on the representation in certain coordinates is not generally accepted. Using the model of Milne, we show that the scale factor of the metric is confused with the time variable.

In 1934, Milne [1,2] introduced a cosmological model based on the principles of special relativity. According to the model, a flat space at first has concentrated matter, which spreads into all directions after an initial process. The redshift of the light emanating from the receding stars has a kinematic origin and is explained by the Doppler effect. The model of Milne has been rejected for physical reasons, yet some cosmologists still show interest [3-9] in it.

The Milne cosmos is regarded as a special case of the Friedman cosmos with the curvature parameter $k=-1$. With a special coordinate transformation the metric should be brought into a flat form to obtain the original description of the Milne cosmos. We critically examine this process here.

We proceed the metric of the Friedman cosmos in case of a negative constant spatial curvature

$$ds^2 = \mathcal{R}^2\left[d\eta^2 + sh^2\eta d\vartheta^2 + sh^2\eta \sin^2\vartheta d\varphi^2\right] - dt^2, \quad \mathcal{R} = \mathcal{R}(t). \tag{1}$$

With the radial coordinate $r = \mathcal{R}\, sh\eta$, we obtain the metric in canonical form

$$ds^2 = \frac{1}{1+\frac{r^2}{\mathcal{R}^2}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - dt^2, \tag{2}$$

from which we read $k=-1$. From (1), we borrow the 4-bein system:

$$\overset{1}{e}_1 = \mathcal{R}, \quad \overset{2}{e}_2 = \mathcal{R}\, sh\eta, \quad \overset{3}{e}_3 = \mathcal{R}\, sh\eta \sin\vartheta, \quad \overset{4}{e}_4 = 1. \tag{3}$$

Using this, we calculate the field strengths from the Ricci-rotation coefficients [11]

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$$A_{mn}{}^{s} = \overset{s}{e}_{i}\, e^{i}_{[n|m]} + g^{sr} g_{mt}\, \overset{t}{e}_{i}\, e^{i}_{[n|r]} + g^{sr} g_{nt}\, \overset{t}{e}_{i}\, e^{i}_{[m|r]} . \tag{4}$$

We then separate them into

$$A_{mn}{}^{s} = B_{mn}{}^{s} + C_{mn}{}^{s} + U_{mn}{}^{s} \tag{5}$$

and with the unit vectors

$$m_m = \{1,0,0,0\}, \quad b_m = \{0,1,0,0\}, \quad c_m = \{0,0,1,0\}, \quad u_m = \{0,0,0,1\} \tag{6}$$

we further split them into

$$B_{mn}{}^{s} = b_m B_n b^s - b_m b_n B^s, \quad C_{mn}{}^{s} = c_m C_n c^s - c_m c_n C^s, \quad U_{mn}{}^{s} = m_m U_n m^s - m_m m_n U^s . \tag{7}$$

Here, B and C are the lateral field quantities and U is the radial field quantity. For the 4th component of all field quantities we get

$$U_4 \overset{*}{=} B_4 \overset{*}{=} C_4 = \frac{1}{\mathcal{R}} \mathcal{R}_{|4} . \tag{8}$$

This means that the expansions in all three spatial directions are the same. Therefore, the curvature scalar has the form

$$u^{s}{}_{||s} = \frac{3}{\mathcal{R}} \mathcal{R}_{|4} \quad .$$

The model expands in free fall because $g_{44} = 1$ is the metric coefficient in (1). Thus, the proper time T is equal to the coordinate time t. The right-hand side of (8) can also be written as

$$\frac{i}{\mathcal{R}} \frac{\partial \mathcal{R}}{\partial T} = \frac{i}{\mathcal{R}} \mathcal{R}^{\cdot} . \tag{9}$$

Further, concerning Milne's ansatz, we put

$$\mathcal{R}^{\cdot} = 1, \tag{10}$$

where 1 is the velocity of light in the natural measuring system. Finally, with (4)-(8) we have

$$\begin{aligned} U_m &= A_{1m}{}^{1} = \{0,0,0,1\}\left(-\frac{i}{\mathcal{R}}\right) \\ B_m &= A_{2m}{}^{2} = \left\{\frac{1}{\mathcal{R}} \mathrm{cth}\eta, 0, 0, -\frac{i}{\mathcal{R}}\right\} = \{\mathrm{ch}\eta, 0, 0, -i\,\mathrm{sh}\eta\}\left(\frac{1}{\mathcal{R}\,\mathrm{sh}\eta}\right) \\ C_m &== A_{3m}{}^{3} = \left\{\frac{1}{\mathcal{R}} \mathrm{cth}\eta, \frac{1}{\mathcal{R}\,\mathrm{sh}\eta} \cot\vartheta, 0, -\frac{i}{\mathcal{R}}\right\} \\ &= \{\mathrm{ch}\eta \sin\vartheta, \cos\vartheta, 0, -i\,\mathrm{sh}\eta \sin\vartheta\}\left(\frac{1}{\mathcal{R}\,\mathrm{sh}\eta \sin\vartheta}\right) \end{aligned} . \tag{11}$$

Here, $\frac{i}{\mathcal{R}}, \frac{1}{\mathcal{R}\,\mathrm{sh}\eta}, \frac{1}{\mathcal{R}\,\mathrm{sh}\eta \sin\vartheta}$ are the curvatures of normal and oblique slices of the surface with the metric (1), which is the base of the model. The relations

$$\underset{1}{U^{s}{}_{||s} + U^{s} U_s = 0}, \quad \underset{2}{B_{m||n} + B_m B_n = 0}, \quad \underset{3}{C_{m||n} + C_m C_n = 0} \tag{12}$$

are formulated with the unit vectors (6) and the graded derivatives

$$U_{m\underset{1}{||}n} = U_{m|n}, \quad B_{m\underset{2}{||}n} = B_{m|n} - U_{nm}{}^{s}B_s, \quad C_{m\underset{2}{||}n} = C_{m|n} - U_{nm}{}^{s}C_s - B_{nm}{}^{s}C_s . \tag{13}$$

They are subequations of the Ricci [11]. Since all the expressions in (12) vanish, we have $R_{mn} = 0$. Therefore the stress-energy-momentum tensor is $T_{mn} = 0$. The cosmos is empty.

The special feature of the Milne model is the simple dependency of the scale factor on time:

$$\Re(t) = t \tag{14}$$

Instead of (9) we have

$$\frac{i}{t}\frac{\partial t}{\partial T} = \frac{i}{t}\frac{\partial t}{\partial t} = \frac{i}{t}$$

and instead of (10) the trivial relation $t^{\bullet} = 1$ which justifies the approach (10) in retrospect. With this and with $r = \Re\, sh\eta$, the simple expressions for the field quantities are obtained as follows:

$$U_m = \{0,0,0,1\}_i\left(-\frac{i}{t}\right), \quad B_m = \left\{\frac{1}{r}ch\eta, 0, 0, -\frac{i}{t}\right\}, \quad C_m = \left\{\frac{1}{r}ch\eta, \frac{1}{r}\cot\vartheta, 0, -\frac{i}{t}\right\} \tag{15}$$

If we also identify the quantity t introduced by (14) with the time variable, it is possible to apply the coordinate transformation

$$r' = t\,sh\eta, \quad t' = t\,ch\eta . \tag{16}$$

Thus, we first obtain

$$dr'^2 - dt'^2 = t^2 d\eta^2 - dt^2 .$$

Now the metric (1) can be written as

$$ds^2 = t^2\left[d\eta^2 + sh^2\eta d\vartheta^2 + sh^2\eta \sin^2\vartheta d\varphi^2\right] - dt^2 \tag{17}$$

and with the above relation as

$$\begin{gathered} ds^2 = dr'^2 + r'^2 d\vartheta^2 + r'^2 \sin^2\vartheta d\varphi^2 - dt'^2 \\ \overset{1'}{e}_{1'} = 1, \quad \overset{2'}{e}_{2'} = r', \quad \overset{3'}{e}_{3'} = r'\sin\vartheta, \quad \overset{4'}{e}_{4'} = 1 \end{gathered} . \tag{18}$$

This is obviously the metric in the flat Minkowski space. It seems that with a coordinate transformation

$$\Lambda_i^{i'} = \begin{pmatrix} tch\eta & -ish\eta \\ tish\eta & ch\eta \end{pmatrix}$$

and the associated pseudo rotation

$$L_m^{m'} = \begin{pmatrix} ch\eta & -ish\eta \\ ish\eta & ch\eta \end{pmatrix} = \begin{pmatrix} \cos i\eta & -\sin i\eta \\ \sin i\eta & \cos i\eta \end{pmatrix}, \quad L_m^{m'} = \overset{m'}{e}_{i'}\Lambda_i^{i'}\overset{i}{e}_m \tag{19}$$

a negatively curved space $(k = -1)$ can be made from a flat space $(k = 0)$.

Using the pseudo rotation (19), the field quantities (11) can be converted into those of a static system. The Ricci-rotation coefficients transform according to

$$'A_{m'n'}{}^{s'} = L^{m\,n\,s'}_{m'n's}\, A_{mn}{}^{s} + 'L_{m'n'}{}^{s'}.$$

The quantity U is the very component of the Ricci-rotation coefficients, which transforms inhomogeneously

$$'U_{m'n'}{}^{s'} = U_{m'n'}{}^{s'} + 'L_{m'n'}{}^{s'}.$$

With

$$\begin{gathered} U_{m'n'}{}^{s'} = h_{m}{}^{s'} U_{n'} - h_{m'n'} U^{s'}, \quad h_{m'n'} = \mathrm{diag}\{1,0,01\} \\ 'L_{m'n'}{}^{s'} = h_{m'}{}^{s'}\, 'L_{n'} - h_{m'n'}\, 'L^{s'}, \quad 'L_{n'} = 'L_{s'n'}{}^{s'} = \left\{ 'L_{4'1'}{}^{4'}, 'L_{1'4'}{}^{1'} \right\} \end{gathered} \tag{20}$$

we finally obtain the simple relations

$$'U_{m'} = U_{m'} + 'L_{m'}\ .$$

With

$$'L_{m'n'}{}^{s'} = L^{s'}_{s} L^{s}_{n'|m'}, \quad 'L_{1'} = 'L_{4'1'}{}^{4'} = -i\eta_{|4'}, \quad 'L_{4'} = 'L_{1'4'}{}^{1'} = i\eta_{|1'}, \tag{21}$$

and with $\eta_{|m'} = L^{m}_{m'} e^{i}_{m} \eta_{|i}$ and (3), we first have

$$'L_{m'} = \{-i\,\mathrm{sh}\eta, 0, 0, \mathrm{ch}\eta\} \frac{i}{t} \tag{22}$$

and from the inhomogeneous transformation law of the Ricci-rotation coefficients

$$U_{m'} = L^{m}_{m'} U_{m} = \{-i\,\mathrm{sh}\eta, 0, 0, \mathrm{ch}\eta\} \left(-\frac{i}{t} \right), \quad 'U_{m'} = U_{m'} + 'L_{m'} = 0\,. \tag{23}$$

In the transformed system, expansion is not experienced; the space would be not only flat but also static.

To resolve this contradiction, we once again deal with the transformation (16) and we revisit (14). Thus, we obtain

$$r' = \mathcal{R}\,\mathrm{sh}\eta, \quad t' = \mathcal{R}\,\mathrm{ch}\eta\,. \tag{24}$$

Recognize that t' is an inappropriate identification with the time variable, we instead write

$$x^{1'} = \mathcal{R}\,\mathrm{sh}\eta, \quad x^{0'} = i\mathcal{R}\,\mathrm{ch}\eta\,, \tag{25}$$

and we obtain with

$$dx^{1'} = \mathrm{sh}\eta d\mathcal{R} + \mathcal{R}\,\mathrm{ch}\eta d\eta, \quad dx^{0'} = i\,\mathrm{ch}\eta d\mathcal{R} + i\mathcal{R}\,\mathrm{sh}\eta d\eta \tag{26}$$

after all

$$dx^{0'2} + dx^{1'2} = \left(i d\mathcal{R} \right)^{2} + \mathcal{R}^{2} d\eta^{2}\,. \tag{27}$$

Now, it can be see that the angle $i\eta$ describes a rotation in the [0',1']-plane of the embedding space and not one in the [1',4']-plane of the physical space. Therefore (19) cannot be a Lorentz transformation that performs the transition from the comoving to the non-comoving system. It should be noted that (25) describes the relationship between the hyperspheric coordinates of the Milne model and the Cartesian coordinates of the flat 5-dimensional embedding space. The complete transformation reads as

$$
\begin{aligned}
x^{3'} &= i\mathcal{R}\sin i\eta \sin\vartheta \sin\varphi = \mathcal{R}\,\mathrm{sh}\eta \sin\vartheta \sin\varphi \\
x^{2'} &= i\mathcal{R}\sin i\eta \sin\vartheta \cos\varphi = \mathcal{R}\,\mathrm{sh}\eta \sin\vartheta \cos\varphi \\
x^{1'} &= i\mathcal{R}\sin i\eta \cos\vartheta = \mathcal{R}\,\mathrm{sh}\eta \cos\vartheta \qquad . \qquad (28) \\
x^{0'} &= i\mathcal{R}\cos i\eta = i\mathcal{R}\,\mathrm{ch}\eta \\
x^{4'} &= it
\end{aligned}
$$

The Milne model with the metric (1) can be expressed geometrically as the surface of a pseudosphere

$$x^{a'}x^{a'} = (i\mathcal{R})^2, \quad a' = 0',1',\ldots,3'$$

with an imaginary radius. This corresponds to the tradition of interpreting cosmological models. Recalling the above-mentioned considerations we can see that the flat metric (18) is obtained. A transformation has been performed into the flat embedding space. The misleading approach (16) was possibly been used first by Walker [10] and has been adopted by other authors.

Conclusions

The attempt to transform the comoving coordinate system of the Milne cosmos into a non-comoving coordinate system has failed because the radius of curvature of the imaginary hypersphere defining the geometry was confused with the time variable. However, we can reject the assertion that the curvature of the space may depend on the representation by pointing out these positions.

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