

EQUIPPING THE MODEL OF MELIA WITH CURVATURE

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Abstract: We compare Melia's $R_h=ct$ model with our Subluminal Model and conclude that both are positively curved and finite. We use the 2nd fundamental forms to formulate the field equation of these models.

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1. INTRODUCTION

In cosmology, there are several competing models. Those that allow for the expansion of the universe are closer to Nature. Besides the static pressure-free Einstein cosmos, the static de Sitter (dS) cosmos deserves consideration. The latter describes a closed curved space in which forces act to drive particles apart in all directions. The dS cosmos can be geometrically interpreted as a pseudo-hypersphere with constant curvature, embedded in a 5-dimensional flat space. The fronts of the diverging particles form an expanding 3-sphere.

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The dS model can be the starting point for an expanding model if the condition $\mathcal{R} = \text{const.}$ is dropped, i.e., the radius \mathcal{R} of the pseudo-hypersphere is assumed to be time-dependent. Then neighboring points will move away from each other. Receding points are located on a 3-sphere, that is, in the space in which we live. This is the basic design of our Subluminal Model.

However, the motions in the dS cosmos and the Subluminal Model are fundamentally different. In the first case, the points move in a static space, in the second case they are at rest, but the space expands, taking the points with it and increasing their distances. The motion follows Hubble's law $v = Hr$, which allows superluminal velocity for large distances of r . This is accepted for some models, where it is assumed that the expansion-induced motion is not a physically relevant motion. We do not want to get involved in considering that interpretation.

We demand that the principle of relativity be adhered to everywhere and at all times. Our Subluminal Model does not allow for superluminal velocities. The name of the model indicates this fact. The recession velocity of galaxies is the greatest at the equator of the pseudo-hypersphere, relative to an observer who defines his position as the pole of the pseudo-hypersphere. The equator is the cosmic horizon.

Models with unbounded r must be open, i.e., flat or negatively curved. They are infinite, and an infinite amount of matter was created in the Big Bang. Melia [1] has proposed a model, which he called the $R_h = ct$ model. He assumed that this model was flat and infinite. In the next section, we will compare Melia's model to our Subluminal Model.

2. MELIA'S MODEL VS. SUBLUMINAL MODEL

Over the past decades, numerous expanding world models have been published, under the name FRW models. Most researchers assume an approach to the metric in comoving coordinates, with spherical coordinates being preferred.

For an expanding universe, the canonical form of the metric can be written as

$$ds^2 = \frac{1}{1 - k \frac{r^2}{\mathcal{R}^2}} dr^2 + r^2 d\Omega^2 + g_{44} dx^4{}^2, \quad dx^4 = i(c) dt. \quad (2.1)$$

Here, k is called the curvature parameter, with the values of $k = (1, 0, -1)$, possibly indicating either a positively curved, flat, or negatively curved universe. Ω contains spherical or hyperbolic angular functions, and \mathcal{R} is a time-dependent variable. A special case is

$$ds^2 = dt^2 - \mathcal{R}^2 (dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2), \quad (2.2)$$

\mathcal{R} being the time-dependent scale factor, r and t coordinates, comoving with the expansion. In this case, the curvature parameter is $k = 0$, and the universe is assumed to be flat and infinite. In several papers [2-4], we outlined that $k = 0$ indicates a universe expanding in free fall. Thus, the metric (2.2) describes a *locally flat* but not a *globally flat* universe.

This view is justified since Lemaître [5] showed that the metric of a positively curved space with $k = 1$ can be transformed into a metric with $k = 0$, describing a system being in free fall. According to Einstein's elevator principle, space appears locally flat to a comoving observer.

Melia also uses the notation in Cartesian coordinates

$$ds^2 = dt^2 - \mathcal{K}^2 \left(dx^1{}^2 + dx^2{}^2 + dx^3{}^2 \right) \quad (2.3)$$

instead of (2.2), apparently to emphasize the flatness and infinity of space. For the scale factor \mathcal{K} he sets $\mathcal{K}(t) = t/t_0$, where t_0 is the time that has passed since the Big Bang.

The lapse function of Melia's metric is $g_{00} = 1$. Thus, no accelerations can be derived from the time-like metrical function. t is the global time, valid for all observers, as well as the proper time for these observers. The immediate consequence is that the universe expands in free fall, but the expansion is acceleration-free. Melia [7] evaluated the astrophysical data from the PLANCK project and showed that they fit his predictions very well. However, the values differ significantly from those predicted by several FRW models.

Melia's model and our Subluminal Model are based on the same metric (2.2). Both are exact solutions of Einstein's field equations, exclude an acceleration of the expansion, and lead to the same EOS. The question is whether these models are not only similar, but identical. Our Subluminal Model is an extension of the dS model and builds on an expanding pseudo-hypersphere. Thus, the model is positively curved and closed. In Hubble's law, $v = Hr$, the distance r is bounded. Its highest value is $r = \mathcal{R}$, the value at the equator of pseudo-hypersphere, where $\mathcal{R} = \mathcal{R}(t)$ is the time-dependent radius of the pseudo-hypersphere. Thus, the model excludes superluminal velocities and their associated unphysical effects. We prefer the notation

$$ds^2 = \mathcal{K}^2 \left(dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2 \right) - dt^2. \quad (2.4)$$

The Friedman equations are

$$\mathcal{R}' = 1, \quad \mathcal{R}'' = 0. \quad (2.5)$$

Therefore, the expansion is linear. The EOS is

$$\mu_0 + 3p = 0. \quad (2.6)$$

To approach the question of how similar the two models are, we rely on a paper by Melia [8], in which he described the field quantities of his model. He represented the field quantities in Riemannian form, i.e., with Christoffel symbols. However, he did not provide the field equations. The calculations are a bit tedious, but are nevertheless provided in Appendix B. It turns out that the spatial components of the Riemann and the Ricci do not vanish, and so the space is globally curved.

The Christoffel symbols are coordinate objects and are only indirectly related to geometric and physical objects. Therefore, we extend them to the Ricci-rotation coefficients, which are composed of the curvature quantities of the surface that represents the model.

First, we read from (2.3) the 4-bein vectors

$$\begin{aligned} \mathbf{e}_1^1 &= \mathcal{K}, & \mathbf{e}_2^2 &= \mathcal{K}, & \mathbf{e}_3^3 &= \mathcal{K}, & \mathbf{e}_4^4 &= 1 \\ \mathbf{e}_1^1 &= \frac{1}{\mathcal{K}}, & \mathbf{e}_2^2 &= \frac{1}{\mathcal{K}}, & \mathbf{e}_3^3 &= \frac{1}{\mathcal{K}}, & \mathbf{e}_4^4 &= 1 \end{aligned} \quad (2.7)$$

and calculate the Ricci-rotation coefficients using Melia's Christoffel symbols Γ :

$$A_{mn}{}^s = \mathbf{e}_m^i \mathbf{e}_n^k \mathbf{e}_j^s \Gamma_{ik}^j + \mathbf{e}_i^s \mathbf{e}_{n|m}^i, \quad (2.8)$$

where m, n, \dots are tetrad indices, and i, k, \dots are coordinate indices. The tetrads represent rods and clocks that measure the structure of space. We use the original Minkowski notation $x^4 = i(c)t$. This has the advantage that $g_{mn} = \delta_{mn}$, $\Phi_m = \Phi^m$.

As an alternative to (2.8), one can calculate the Ricci-rotation coefficients directly from (2.7).

$$A_{14}{}^1 = -\dot{e}_{14}^1 = \frac{1}{\mathcal{R}} \dot{\mathcal{R}}_{14} = -i \frac{\dot{\mathcal{R}}}{\mathcal{R}}. \quad (2.9)$$

Here, the overdot indicates the derivative with respect to global time. Introducing $\mathcal{R} = \mathcal{K} \mathcal{R}_0$ with \mathcal{R}_0 as a constant, we get with (2.5)

$$A_{14}{}^1 = -\frac{i}{\mathcal{R}} = D_{11}. \quad (2.10)$$

Finally, we have

$$D_{11} = D_{22} = D_{33} = -\frac{i}{\mathcal{R}}, \quad D_{mn} = -\frac{i}{\mathcal{R}} g_{mn} \quad (2.11)$$

The D_{mn} are the 2nd fundamental forms of the expanding 3-surface which we intend to discuss, and g_{mn} is the 3-dimensional spatial metric¹.

Integrating the first equation of (2.5), we get

$$\mathcal{R} = (c)t. \quad (2.12)$$

Rescaling $t \rightarrow t/t_0$, we get Melia's scale factor. Looking at our Subluminal Model, then \mathcal{R} is the radius of our universe. If η is the polar angle on the pseudo-hypersphere and $R = \mathcal{R} \sin \eta$ the radial coordinate, we have at the equator $R_h = \mathcal{R} = (c)t$. This is Melia's basic equation.

To proceed with our derivation of field equations, we use the general formulae of the 2nd fundamental forms provided in Mathematical Appendix A. Melia investigates the metric (2.3), using Cartesian coordinates. In this case, the quantity 'A vanishes and the relation (A.2) is reduced to

$$A_{mn}{}^s = D_m{}^s u_n - D_{mn} u^s. \quad (2.13)$$

(A.4) shows that the 3-dimensional part of the Riemann does not vanish and that the geometry is positively curved. We have

$$R_{\gamma\alpha\beta\delta} = D_{\alpha\delta} D_{\beta\gamma} - D_{\alpha\beta} D_{\delta\gamma}, \quad \alpha = 1, 2, 3. \quad (2.14)$$

These are Gauss' equations for embeddings of class one, i.e., for embedding a surface M_n in a space M_{n+1} .

The relation (A. 4) of Appendix A also contains the Codazzi equations

$$D_{[\alpha}{}^\delta{}_{\wedge\gamma]} = 0 \quad (2.15)$$

¹ The reader who is not familiar with Minkowski notation will notice that the factor i drops out of all field equations.

which are trivially satisfied because \mathcal{R} is a spatial constant. Note also that with Melia's notation, the spatial covariant derivatives reduce to the ordinary partial derivatives: $\Phi_{m\wedge n} = \Phi_{m|n}$.

Of the Ricci only remains:

$$\begin{aligned} R_{\alpha\beta} &= -D_{\alpha\beta|4} - D_{\alpha\beta} D_{\gamma}^{\gamma} \\ R_{44} &= -D_{\alpha}^{\alpha}|_4 - D_{\alpha\beta} D^{\alpha\beta} \\ R &= -2D_{\alpha}^{\alpha}|_4 - D_{\alpha\beta} D^{\alpha\beta} - D_{\alpha}^{\alpha} D_{\beta}^{\beta} \end{aligned} \quad (2.16)$$

With (2.11), we obtain

$$\begin{aligned} R_{\alpha\beta} &= \frac{2}{\mathcal{R}^2} g_{\alpha\beta}, \quad R_{44} = 0, \quad R = \frac{6}{\mathcal{R}^2} \\ G_{\alpha\beta} &= -\frac{1}{\mathcal{R}^2} g_{\alpha\beta} = \kappa p g_{\alpha\beta}, \quad G_{44} = -\frac{3}{\mathcal{R}^2} = -\kappa \mu_0 \end{aligned} \quad (2.17)$$

Finally,

$$\kappa p = -\frac{1}{\mathcal{R}^2}, \quad \kappa \mu_0 = \frac{3}{\mathcal{R}^2}, \quad \mu_0 + 3p = 0. \quad (2.18)$$

The first two relations are missing in Melia's papers.

In our paper 'Subluminal Model', we started with the metric of form (2.2). In this scenario the quantity 'A does not vanish. Since the space is locally flat, all subequations of Einstein's field equations with 'A drop out and we get the same result as in (2.17).

In the end, let us take another look at the equation (A.4) in Appendix A. It contains the relation

$$R_{4mn}{}^4 = -2D_{[m\cdot n\cdot\wedge 4]} + 2D_{[4}^s u_{m]} D_{sn} = -D_{m\wedge n|4} - D_m^s D_{ns}$$

which is zero for the Subluminal Model and does not contribute to the Riemann. In contrast, for a static universe, we have $\mathcal{R} = \text{const.}$ and the first term in the above equation vanishes. Thus, an additional term appears in the Ricci:

$$R_{\alpha\beta} = R_{\gamma\alpha\beta}{}^{\gamma} + R_{4\alpha\beta}{}^4 = \frac{2}{\mathcal{R}^2} g_{\alpha\beta} + \frac{1}{\mathcal{R}^2} g_{\alpha\beta} = \frac{3}{\mathcal{R}^2} g_{\alpha\beta}.$$

Further, we have

$$R_{44} = -D_s^s|_4 - D_{rs} D^{rs} = -D_{rs} D^{rs} = \frac{3}{\mathcal{R}^2}.$$

For the Ricci scalar, we get the following:

$$R = R_{\alpha}^{\alpha} + R_4{}^4 = \frac{12}{\mathcal{R}^2},$$

and finally, we calculate the Einstein tensor

$$G_{\alpha\beta} = -\frac{3}{\mathcal{R}^2} g_{\alpha\beta} = \kappa p g_{\alpha\beta}, \quad G_{44} = -\frac{3}{\mathcal{R}^2} = -\kappa \mu_0.$$

With the condition $\mathcal{R} = \text{const.}$, we obtain the dS model from the Subluminal Model. The dS model is represented by a pseudo-hypersphere with a constant radius. We obtain the typical equations

$$\kappa\rho = -\frac{3}{\mathcal{R}^2}, \quad \kappa\mu_0 = \frac{3}{\mathcal{R}^2}, \quad \rho + \mu_0 = 0,$$

well-known from this model. This is less surprising because we derived our Subluminal Model from the dS model by dropping the condition $\mathcal{R} = \text{const.}$.

3. CONCLUSIONS

We analyzed Melia's $R_h=ct$ model using the methodology of the 2nd fundamental forms. We did this in the most general form, i.e., in terms of both the invariant tetrad formalism, and in Melia's coordinates. We calculated the field equations of the model using the results, Melia presented in his papers. We showed that this model actually describes an expanding 3-sphere. Finally, Melia's model and our Subluminal Model are on an equal footing. Contrary to Melia's claim, his model is spatially locally flat but globally curved. This issue can be verified by referring to his recently published results.

4. MATHEMATICAL APPENDIX A

This appendix presents the mathematical structure of expanding models using the 2nd fundamental forms of surface theory. We assume that the models discussed, contain expanding 3-spheres, whose rigging vectors

$$u_m = \{0,0,0,1\}$$

are time-like and perpendicular to the 3-sphere, i.e., perpendicular to its tangents. We define a symmetric spatial quantity as follows:

$$D_{mn} = u_{m||n}, \quad D_{[mn]} = 0, \quad D_{mn}u^n = 0. \quad (\text{A.1})$$

Obviously, this quantity is part of the Ricci-rotation coefficients A :

$$u_{m||n} = u_{m|n} - A_{nm}{}^s u_s = -A_{nm}{}^4 = D_{mn}.$$

By separating this quantity from the Ricci-rotation coefficients, we obtain

$$A_{mn}{}^s = {}^s A_{mn} + D_{mn}{}^s, \quad D_{mn}{}^s = D_m{}^s u_n - D_{mn} u^s, \quad D_{m(ns)} = 0. \quad (\text{A.2})$$

Here, the ${}^s A$ are some spatial components of the Ricci-rotation coefficients. Performing this decomposition in the Riemann

$$R_{rmn}{}^s = 2 \left[A_{[m \cdot n \cdot |r]}{}^s + A_{[m \cdot n \cdot}{}^t A_{r]t}{}^s + A_{[mr]}{}^t A_{tn}{}^s \right], \quad (\text{A.3})$$

we obtain

$$\begin{aligned} R_{rmn}{}^s = & {}^s R_{rmn} + 2u_{[r} \left[{}^s A_{m]n}{}^t u^t + D_m{}^t {}^s A_{tn} \right] \\ & + 2 \left[u_n D_{[m}{}^s{}_{\cdot r]} - D_{[m \cdot n \cdot \wedge r]} u^s + u_n u_{[r} D_m{}^t D_t{}^s + u^s u_{[m} D_r{}^t D_{tn} \right] + 2D_{[m}{}^s D_{r]n}. \end{aligned} \quad (\text{A.4})$$

The quantity ${}^s R_{rmn}$ is written with ${}^s A$ analogous to (A.3). Here, the derivative with respect to ${}^s A$ is defined by

$$\Phi_{m \wedge n} = \Phi_{m|n} - {}^s A_{nm}{}^s \Phi_s, \quad u_{m \wedge n} = 0.$$

By contracting the Riemann, we get the Ricci tensor

$$R_{mn} = {}^i R_{mn} - u_m \left[{}^i A_{sn}^s u^t + D_s^t {}^i A_{tn}^s \right] + 2u_n D_{[m}^s u^s] - D_{mn} u^s - D_{mn} D_s^s - u_m u_n D_{rs} D^{rs}. \quad (\text{A.5})$$

By once again contracting, we obtain the Ricci scalar

$$R = {}^i R - 2D_n^s u^s - D_{rs} D^{rs} - D_n^n D_s^s. \quad (\text{A.6})$$

When applied to the models discussed, these formulae are drastically simplified. More on this topic can be found in [9,10,11].

5. MATHEMATICAL APPENDIX B

In this appendix, we make up for Melia's missing calculations. We keep his field quantities in coordinate notation. In order not to stumble over the factor c during the calculation, we use the natural measure system with $c = 1$. The contents of Melia's field quantities are not changed by this restriction.

Melia's field quantities with $x^0 = t$ are:

$$\Gamma_{11}^0 = \Gamma_{22}^0 = \Gamma_{33}^0 = \frac{t}{t_0^2}, \quad \Gamma_{01}^1 = \Gamma_{02}^2 = \Gamma_{03}^3 = \frac{1}{t}. \quad (\text{B.1})$$

Written in compact form:

$$\Gamma_{\alpha\beta}^0 = \frac{t}{t_0^2} \delta_{\alpha\beta}, \quad \Gamma_{0\beta}^\gamma = \frac{1}{t} \delta_\beta^\gamma, \quad \alpha = 1,2,3. \quad (\text{B.2})$$

The Riemann is:

$$R_{jki}{}^l = 2 \left[\Gamma_{i[kj]}^l + \Gamma_{i[k}^g \Gamma_{j]g}^l \right], \quad i = 1,2,\dots,4$$

and its spatial components

$$R_{\gamma\alpha\beta}{}^\delta = \Gamma_{\beta\alpha\gamma}^\delta - \Gamma_{\beta\gamma\alpha}^\delta + \Gamma_{\beta\alpha}^0 \Gamma_{\gamma 0}^\delta - \Gamma_{\beta\gamma}^0 \Gamma_{\alpha 0}^\delta = \Gamma_{\beta\alpha}^0 \Gamma_{\gamma 0}^\delta - \Gamma_{\beta\gamma}^0 \Gamma_{\alpha 0}^\delta.$$

Inserting (B.2), we obtain the *Gauss equation*

$$R_{\gamma\alpha\beta}{}^\delta = \frac{1}{t^2} \left[\delta_{\alpha\beta} \delta_\gamma^\delta - \delta_\alpha^\delta \delta_{\beta\gamma} \right]. \quad (\text{B.3})$$

Since the spatial components of the Riemann do not vanish, the geometry is globally curved. The mixed components are:

$$R_{0ki}{}^0 = \Gamma_{ik|0}^0 - \Gamma_{i0|k}^0 + \Gamma_{ik}^g \Gamma_{0g}^0 - \Gamma_{i0}^g \Gamma_{kg}^0 = 0.$$

Thus, we obtain the contraction of the Riemann

$$R_{ki} = R_{\gamma ki}{}^\gamma + R_{0ki}{}^0 = R_{\gamma ki}{}^\gamma.$$

Moreover, we have

$$R_{00} = R_{\gamma 00}{}^\gamma = \Gamma_{00\gamma}^\gamma - \Gamma_{0\gamma|0}^\gamma + \Gamma_{00}^g \Gamma_{\gamma g}^\gamma - \Gamma_{0\gamma}^g \Gamma_{0g}^\gamma = 0, \quad R_{0\beta} = 0. \quad (\text{B.4})$$

Now, by contraction of (B.3), we are able to calculate the Ricci as follows:

$$R_{\alpha\beta} = \frac{1}{t_0^2} \left[3\delta_{\alpha\beta} - \delta_{\alpha\beta} \right] = \frac{2}{t_0^2} \delta_{\alpha\beta}.$$

This is a coordinate object. We need to measure its components with rods. Now, μ and ν are triad indices. We strangle the coordinate indices² using

$$e_1^1 = e_2^2 = e_3^3 = \frac{t_0}{t}, \quad R_{\mu\nu} = e_{\mu}^{\alpha} e_{\nu}^{\beta} R_{\alpha\beta}$$

and finally obtain

$$R_{\mu\nu} = \frac{2}{t^2} \delta_{\mu\nu}, \quad R = \frac{6}{t^2}, \quad G_{\mu\nu} = -\frac{1}{t^2} \delta_{\mu\nu} = \kappa p \delta_{\mu\nu}, \quad G_{00} = -\frac{3}{t^2} = -\kappa \mu_0 \quad (\text{B.5})$$

Thus, pressure and mass density

$$\kappa p = -\frac{1}{t^2}, \quad \kappa \mu_0 = \frac{3}{t^2}$$

are geometrically defined quantities, and the EOS is

$$\mu_0 + 3p = 0.$$

Substituting (2.12) into these relations, gives the results (2.18) of the Subluminal Model. The geometry of this model is based on a pseudo-hypersphere, and the universe is positively curved.

6. REFERENCES

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² To avoid triads, we can use $R_{\alpha}^{\beta} = R_{\alpha\gamma} g^{\gamma\beta}$.