ARG Austrian Reports on Gravitation

ARG-2023-01

COVARIANT VARIATIONAL PRINCIPLE IN THE THEORY OF GRAVITATION

Rainer Burghardt*

This paper was published as 'Kovariantes Variationsprinzip in der Gravitationstheorie' in Annalen der Physik 44, 340, 1987. It was an attempt to continue H.-J. Treder's work on the coordinate invariant theory of gravitation. Treder was president of the German Academy of Sciences. He was my mentor and passed away in 2008.

Keywords: Variation principle, Lorentz space, Lagrange function, holonomy, anholonomy

Abstract: Gravitation theory treated in Treder's version of Lorentz space is supplemented by a coordinate invariant variation principle.

The advantage of representing gravitational physics using the Lorentz space, spanned by local tetrads instead of the Einstein space covered by coordinates, has been highlighted several times [1-5]. The Lorentz space as a family of location-dependent orthogonal 4-bein system allows index-wise space-time splitting. Thus, one can also separate physically interpretable variables from complicated complexes. Moreover, it is always assumed that exclusively covariant derivatives are used, and a 3-dimensional equivalent to these derivatives is defined.

The field equations and their coordinate-invariant decompositions have been dealt with in detail in the literature; however, their derivation from a variational principle using the above-mentioned principles has not yet been tackled.

Einstein's method uses the metric g_{ik} and its derivatives as variables, according to which the fundamental invariant $\mathfrak{G}$ is varied, or the g_{ik} and a combination of the derivatives, the Christoffel symbols Γ^{j}_{ik}. However, if the g and Γ are considered to be independent of each other, one obtains the field equations much more quickly. This latter method is attributed to Palatini [6-8].

The metric and the Christoffel symbols are not apparently physically interpretable quantities; they are elements of the basic geometric structure from which the gravitational phenomena originate. The gravitational variables themselves are constructed from local 4-bein systems, which, however, 'nestle' to the basic geometric structure. Thus, it is reasonable that changes in geometry, i.e., deformations of the underlying space, also affect those local systems that we understand as a direct manifestation of gravitational

* e-mail: arg@aon.at, arg@arg.or.at, home page: http://arg.or.at/

physics. Therefore, it is to be expected that the field equations can be derived not only with Einstein's method but also with a method that corresponds to the induced changes in the local systems.

Of the two types of systems, the holonomic system $\overset{a}{e}_i$ is closely linked to geometry and is always chosen such that the reciprocal vectors $\underset{a}{e}^i$ (the index a = 1,...,4 numbers them) are tangent to a net of lines (a). If this is identical to the coordinate system (i), then one obtains: $\underset{a}{e}^i = \delta^i_a$. The anholonomic system $\overset{m}{e}_i$ is more closely related to physics. It is chosen in such that usually only one of the four reciprocal vectors $\underset{m}{e}^i$, $(m = 1,...4)$ is tangent to the coordinate net, and the others are orthogonal to this and to one another. There have been several possibilities of choice and their usefulness in gravity models discussed in papers [4]. An anholonomic transformation mediates between the two systems as follows:

$$\overset{m}{e}_i = A^m_a \overset{a}{e}_i, \quad \overset{a}{e}_i = A^a_m \overset{m}{e}_i. \tag{1}$$

This has proved quite useful for implementing the variational principle with the quantities of A^m_a or A^a_m, and in further aggregates built with their first derivatives, i.e., $H_{ab}{}^c$, or $H_{mn}{}^s$, respectively. The A^a_m can be interpreted as gravitational potentials, the 3-rank quantities as gravitational field strengths, while the latter ones operate as coefficients of the holonomic and anholonomic connexion[1]:

$$\begin{aligned} &V_{\underset{H}{a||b}} = V_{a|b} - H_{ba}{}^c V_c, \quad V_{\underset{A}{m||n}} = V_{m|n} - A_{nm}{}^s V_s, \\ &V_{\underset{A}{m||n}} = A^{ab}_{mn} V_{\underset{H}{a||b}}, \quad H_{abc} = g_{mn} A^m_{\{(a} A^n_{\underset{A}{b)||c\}}}, \quad A_{mns} = -g_{ab} A^a_{\{m} A^b_{\underset{H}{[n||s]\}}}. \end{aligned} \tag{2}$$

While performing a variation in the presence of a gravitational field, difficulties may arise which are unknown in other field theories. First, one is tempted to carry out the procedures with the usual partial derivatives. On the one hand, we are used to this procedure. On the other hand, there is a need for it, for example, in the treatment of surface integrals. However, expressions with ordinary partial derivatives are not covariant and do not describe any physical facts. However, it is quite easy to show that the following Lagrangians

$$\mathcal{L}' = \mathcal{L}'\left(\varphi_a, \varphi_{a|b}\right), \quad \mathcal{L} = \mathcal{L}\left(\varphi_a, \varphi_{\underset{H}{a||b}}\right) \tag{3}$$

lead to the same Lagrange equations:

$$\frac{\partial \mathcal{L}'}{\partial \varphi_a} - \left(\frac{\partial \mathcal{L}'}{\partial \varphi_{a|b}}\right)_{|b} = \frac{\partial \mathcal{L}}{\partial \varphi_a} - \left(\frac{\partial \mathcal{L}}{\partial \varphi_{a||b}}\right)_{||b}. \tag{4}$$

The same applies to the anholonomic representation. In this manner, every matter field with a covariant set of formulae can be connected to the theory of gravitation. To get the covariant Lagrange equation, a partial integration has to be carried out, and a surface integral must be omitted. If one proceeds carefully, this procedure can also be mastered in the covariant representation. The integral $\int \underset{A}{d}^4 x$ describes a volume which is bounded by four normal vectors dx^m. Thus, the following arises:

---

[1] The curly brackets denote Christoffel symmetry.

$$\int \underset{A}{d}^4 x = \int A \underset{H}{d}^4 x, \quad A = \det A^m_a . \tag{5}$$

The elements of the right integral are bounded by four holonomic, non-orthogonal vectors dx^a. The holonomy allows the integration to be executed. If one faces two tensors $P^{\cdots}$, $Q_{\cdots}$, the partial integration yields the following:

$$P^{\cdots a} Q_{\cdots \underset{H}{||a}} = -P^{\cdots a}{}_{\underset{H}{||a}} Q_{\cdots}, \quad P^{\cdots m} Q_{\cdots \underset{A}{||m}} = -P^{\cdots m}{}_{\underset{A}{||m}} Q_{\cdots}, \tag{6}$$

where, in both cases, the surface integral

$$\int \left(A P^{\cdots a} Q_{\cdots} \right)_{|a} \underset{H}{d}^4 x \tag{7}$$

is omitted. However, far more difficulties are caused by the non-interchangeability of the variation and the covariant differential operators:

$$\left[\delta, \underset{H}{D} \right] \neq 0, \quad \left[\delta, \underset{A}{D} \right] \neq 0 . \tag{8}$$

Thus, from Eq. (3) one first arrives at

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi_a} \delta \varphi_a + \frac{\partial \mathcal{L}}{\partial \varphi_{a \underset{H}{||b}}} \delta \left(\varphi_{a \underset{H}{||b}} \right).$$

But because Eq. (8) we do not obtain Eq. (4), if the variation affects the gravitational field itself, i.e., $\delta H_{ab}{}^c \neq 0$. For the derivation of the gravitational equations, one would have to take as variables the following:

$$A^m_a, \, A^m_{a \underset{H}{||b}}, \quad A^a_m, \, A^a_{m \underset{A}{||n}} . \tag{9}$$

However, to circumvent the outlined difficulties instead of taking (9) the following variables are used:

$$A^m_a, \quad H_{ab}{}^c, \quad \text{or} \quad A^a_m, \quad A_{mn}{}^s . \tag{10}$$

Correspondingly, the Lagrange equations change, as described by Eq. (4). The implementation of the variation leads to Einstein's field equations in both the holonomic and the anholonomic cases.

We derive the variation itself from a deformation of the geometry. This is created from a point transformation

$$\bar{x}^a = e^{\theta} x^a, \quad \theta = \theta^b \partial_b \tag{11}$$

with infinitesimal quantities θ^b. A variable φ^a changes accordingly:

$$\varphi^a \to \varphi^a - \theta^b \partial_b \varphi^a . \tag{12}$$

However, we consider the point-transformed quantity from the point of view of the trailed curve mesh, which arises from

$$x^{a'} = \delta^{a'}_a \left(x^a - \theta^a \right). \tag{13}$$

The difference in φ is the Lie differential, which we denote by $\delta\varphi$. Applied to the gravitational quantities, we get the aggregates as follows[2]:

[2] The curly brackets mean Christoffel symmetry: $\theta_{\{abc\}} = \theta_{bac} + \theta_{cab} - \theta_{abc}$.

$$A^a_m \delta A^m_b = \theta^a{}_b, \quad \theta_{ab} = \theta_{a \underset{H}{||} b} - A_{cab}\theta^c,$$
$$A^a_n \delta A^m_a = \theta^m{}_n, \quad \theta_{mn} = \theta_{m \underset{A}{||} n} - A_{smn}\theta^s, \tag{14}$$
$$g_{ec}\delta H_{ab}{}^c = \theta_{\{(ab)\underset{H}{||}e\}}, \quad A^a_m \delta A_{ans} = \theta_{\{m[n\underset{A}{||}s\}}.$$

The θ_{ab}, θ_{mn} describe a twist deformation of the 4-bein systems; its symmetrical part

$$\theta_{(ab)} = \theta_{(a\underset{H}{||}b)}, \theta_{(mn)} = \theta_{(m\underset{A}{||}n)}$$

is merely a deformation. Only the symmetrical part of the point transformation occurs in the Lagrange equation. Lastly, Einstein's field equations are symmetric in their two indices:

$$\begin{aligned} \delta\mathfrak{G}_H &= -2\left(\mathcal{R}_{ab} - \frac{1}{2}g_{ab}\mathcal{R}\right)\theta^{(ab)}, \quad \mathfrak{G}_H = A\left(H_{abc}H^{cba} - H_e{}^{ec}H_{dc}{}^d\right) \\ \delta\mathfrak{G}_A &= -2\left(\mathcal{R}_{mn} - \frac{1}{2}g_{mn}\mathcal{R}\right)\theta^{(mn)}, \quad \mathfrak{G}_A = A\left(A_{rs}{}^r A_n{}^{sn} - A_{mns}A^{snm}\right) \end{aligned}. \tag{15}$$

The antisymmetric part $\varepsilon_{mn} = \theta_{[mn]}$ can be interpreted as a Lorentz rotation, and it leaves the Lagrange function unchanged:

$$\delta_\varepsilon \mathfrak{G}_A = 0. \tag{16}$$

Thus, the fundamental invariant of the theory of gravitation is Lorentz invariant. The same must also apply to an extended Lagrangian that contains matter fields. Remarkably, Utiyama derived the existence of the three-rank quantity A in the opposite way, i.e., from the requirement of the Lorentz invariance of the Lagrangian of matter fields. Extending the ideas of Yang and Mills, such fields must be introduced as compensating auxiliary fields when gravitation acts on matter. Numerous authors have dealt with the extension of the Lorentz group S0(3,1) to the Poincaré group, e.g., Kibble [9] and Hehl et al. [10]. Deriving the gravitational equations directly from a gauge principle has also been the subject of many investigations. Cho [11] used the abelian translation group T_4, which is quite close to our ideas, but the process requires the decomposition of the 4-bein field into a flat and a non-trivial part:

$$\overset{m}{e}{}_i = \delta^m_i + B^m_i .$$

The latter is the gauge field, which transforms inhomogeneously under T_4.

However, we find that the quantities A^a_m have the desired inhomogeneous transformation behavior under the Einstein group. From Eq. (14) it follows:

$$A^a_m \delta A^s_a = \theta^s{}_{|m} + 2A_{[mn]}{}^s\theta^n \quad \text{or} \quad \delta A^s_a = \theta^s{}_{|a} + 2A_{[mn]}{}^s A^m_a \theta^n . \tag{17}$$

These relations are reminiscent of the gauge transformation of boson fields of the electroweak interaction, since the following is true:

$$\left[\partial_m, \partial_n\right] = 2A_{[mn]}{}^s \partial_s . \tag{18}$$

The difficulties in presenting the theory of gravitation as a gauge theory probably lie in the vagueness of the separation between the internal group space and the external visual space. The latter is certainly the set of local spaces spanned by the anholonomic vectors $\underset{m}{e}{}^i$; the index sequence $\{m,n,...\}$ designates these spaces. As an internal space with respect to GL(4, R), one has to take the Riemannian V_4 with the index sequence

$\{a,\ b,...\}$. The Lie operation also acts on these indices. However, taking relation (1), terms such as external and internal can be mixed up at any time. In the papers of Cho, first an internal space is constructed with the instrument of fiber bundle geometry. Finally, this space is again equated with the external one by the identification of its elements with those of conventional Riemannian geometry. Derbes [12] pointed out the coexistence of the Einstein and Lorentz groups. We ourselves do not seek any group-theoretical deepening in this paper but only want to establish relations to familiar principles.

I am grateful to Prof. Dr. H.-J. Treder for numerous comments on this paper.

References

[1] Treder H.-J., Borzeszkowski H.-H. V., Int. Journ. Theor. Phys. **4**, 295, 1971

[2] Brauer H.-J., Int. Journ. Theor. Phys. **4,** 243, 1971

[3] Treder H.-J., Int. Journ. Theor. Phys. **3**, 23, 1970

[4] Burghardt R., Acta Phys. Austr. **51**, 1, 1979 and **52,** 61, 1980

[5] Burghardt R., Ann. d. Phys. **41,** 397, 1984

[6] Ellis G. F. R., Journ. Math. Phys. **8**, 1171, 1967

[7] Hehl F. W., Kerlik G. D., GRG **9,** 691, 1978

[8] Hehl F. W., Lord. E. A., Smalley L. L., GRG **18**, 11, 1981

[9] Kibble T. W. B., Journ. Math. Phys. **2,** 212, 1961

[10] Hehl F. W. et al., Rev. Mod. Phys. **48,** 393, 1976

[11] Cho Y. M., Phys. Rev. **D 14**, 2521, 3241 and 3335, 1976

[12] Derbes D., Nuov. Cim. **45**, 31, 1978.