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# TRANSFORMATIONS IN DE SITTER AND LANCZOS MODELS III.

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Contents



Abstract: In previous papers [1,2] we have studied transformations between comoving and non-comoving systems for the cosmological models of the de Sitter family. Transformations between expanding coordinates and static coordinates are well known in the literature. Based on these coordinate transformations we derive Lorentz transformations between comoving and non-comoving reference systems. We borrow the relative velocity between these reference systems from a Lorentz transformation. In the present paper, we start from the static de Sitter model, but we drop the condition of the constancy of the spatial curvature. Thus we obtain a cosmological model which includes mass density and pressure and which is an exact solution of Einstein's field equations. The model is related to the Friedman cosmos and the Melia cosmos.

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# 1. INTRODUCTION

In this paper we present an exact solution of Einstein's field equations for a cosmological model which includes pressure. It is not our intention to propose a new cosmological model which is close to Nature. We want to provide some mathematical methods for astrophysicists, who want to build a new model. In the present case we start from the static de Sitter universe. We extend this model in such a way that the basic geometric structure is time-dependent. The model is positively curved and closed. In Sec. 2, we revise the de Sitter universe. We explain the tetrad method, we calculate the Ricci-rotation coefficients, and we note the Lorentz transformation, which transforms the non-comoving 4-bein system of the static dS model into the comoving 4-bein of an expanding observer system. In Sec. 3 we allow the curvature of the dS cosmos to be time-dependent and thus we obtain an expanding model with pressure. In Sec. 4. we derive the field quantities for the non-comoving system with a Lorentz transformation and we realize that with this transformation we do not return to the static dS model. In Sec. 5 we work out some features of the model. In particular, we show that the recession velocity of galaxies cannot exceed the speed of light. Attention is drawn to the similarity of this model to the model of Melia.

# 2. PRELIMINARY REMARKS

In [1] we have assumed that the de Sitter cosmos need not be interpreted as an expanding model. The coordinate grid obtained by Lemaître with the help of a suitable transformation expands from an arbitrarily chosen point, whereby the space itself remains unchanged. This coordinate system can be assigned to an observer system which consists of four orthogonal unit vectors (tetrads) and which joins the motion of the coordinate system. From the metric of the static version of the dS cosmos the tetrads of the static system can be read. They are connected with the comoving one via a Lorentz transformation, from which we read the relative velocity of the observer. The method can be assessed more generally in the following way: If is $\Lambda^{i'}_{i} = x^{i'}{}_{|i}$ the matrix of the coordinate transformation[1] then

$$L^{m'}_{m} = \overset{m'}{e}{}_{i'} \Lambda^{i'}_{i} \underset{m}{e}{}^{i} \tag{2.1}$$

is the associated Lorentz transformation with

$$L^{m'}_{m} = \begin{pmatrix} \alpha & & & -i\alpha v \\ & 1 & & \\ & & 1 & \\ i\alpha v & & & \alpha \end{pmatrix}. \tag{2.2}$$

$\alpha$ is the Lorentz factor and v is the relative velocity of the observers. The 4-bein systems of the comoving and the non-comoving systems are

[1] $dx^4 = i dt, \quad dx^{4'} = i dt'$

$$\overset{1'}{e}_{1'} = e^{\psi'}, \quad \overset{2'}{e}_{2'} = r, \quad \overset{3'}{e}_{3'} = r\sin\vartheta, \quad \overset{4'}{e}_{4'} = 1$$

$$\overset{1}{e}_{1} = \alpha, \quad \overset{2}{e}_{2} = r, \quad \overset{3}{e}_{3} = r\sin\vartheta, \quad \overset{4}{e}_{4} = a\,. \tag{2.3}$$

$$a = 1/\alpha = \sqrt{1 - r^2/\mathcal{R}_0^2}$$

Herein $r, \vartheta, \varphi$ are polar coordinates and $\mathcal{R}_0$ the time-constant radius of the pseudo-hyper sphere which is responsible for the geometric framework of the de Sitter geometry.

With (2.3) and

$$A_{mn}{}^{s} = \overset{s}{e}_{i}\, e^{i}_{[n|m]} + g^{sr} g_{mt}\, \overset{t}{e}_{i}\, e^{i}_{[n|r]} + g^{sr} g_{nt}\, \overset{t}{e}_{i}\, e^{i}_{[m|r]} \tag{2.4}$$

we can calculate the Ricci-rotation coefficients which we decompose according to

$$A_{mn}{}^{s} = B_{mn}{}^{s} + C_{mn}{}^{s} + U_{mn}{}^{s}\,. \tag{2.5}$$

With the unit vectors

$$m_m = \{1,0,0,0\}, \quad b_m = \{0,1,0,0\}, \quad c_m = \{0,0,1,0\}, \quad u_m = \{0,0,0,1\} \tag{2.6}$$

we further split (2.5) into

$$B_{mn}{}^{s} = b_m B_n b^s - b_m b_n B^s, \quad C_{mn}{}^{s} = c_m C_n c^s - c_m c_n C^s, \quad U_{mn}{}^{s} = u_m U_n u^s - u_m u_n U^s\,. \tag{2.7}$$

B and C are the lateral field quantities, and U the radial field quantity

$$B_m = \left\{\frac{a}{r},0,0,0\right\}, \quad C_m = \left\{\frac{a}{r},\frac{1}{r}\cot\vartheta,0,0\right\}, \quad U_m = \left\{-\alpha v\frac{1}{\mathcal{R}_0},0,0,0\right\}, \quad v = \frac{r}{\mathcal{R}_0}\,. \tag{2.8}$$

With these quantities and using the graded derivatives

$$U_{n\underset{1}{||}m} = U_{n|m}, \quad B_{n\underset{2}{||}m} = B_{n|m} - U_{mn}{}^{s} B_s, \quad C_{n\underset{3}{||}m} = C_{n|m} - U_{mn}{}^{s} C_s - B_{mn}{}^{s} C_s \tag{2.9}$$

we calculate the Ricci

$$R_{mn} = A_{mn}{}^{s}{}_{|s} - A_{n|m} - A_{rm}{}^{s} A_{sn}{}^{r} + A_{mn}{}^{s} A_s, \quad A_n = A_{rn}{}^{r}$$

$$\begin{aligned} R_{mn} = & -\left[U^{s}{}_{\underset{1}{||}s} + U^s U_s\right] h_{mn} \\ & -\left[B_{n\underset{2}{||}m} + B_n B_m\right] - b_n b_m \left[B^{s}{}_{\underset{2}{||}s} + B^s B_s\right] \\ & -\left[C_{n\underset{3}{||}m} + C_n C_m\right] - c_n c_m \left[C^{s}{}_{\underset{3}{||}s} + C^s C_s\right] \end{aligned} \tag{2.10}$$

$$-\frac{1}{2}R = \left[U^{s}{}_{\underset{1}{||}s} + U^s U_s\right] + \left[B^{s}{}_{\underset{2}{||}s} + B^s B_s\right] + \left[C^{s}{}_{\underset{3}{||}s} + C^s C_s\right]$$

$h_{mn} = \mathrm{diag}(1,0,0,1)$ is a submatrix of the tetrad metric $g_{mn} = \mathrm{diag}(1,1,1,1)$.

# 3. THE MODEL WITH PRESSURE

The model of Friedman is pressure-free and because of its simplicity it does not seem to explain the cosmic reality sufficiently. An extension of the stress-energy momentum tensor by pressure components requires the solution of a further equation. For its determination parameters are necessary, which one has to determine by observations. We want to investigate whether it is possible to develop an expanding cosmological model which draws the pressure as well as the mass density from the geometric structures of the model. We also require the model to be an exact solution of Einstein's field equations. The field equations should describe both the physics of an observer who comoves with the expansion and of an observer who does not participate in this motion. Between the two observer systems a Lorentz transformation should mediate. The relative velocity contained in this Lorentz transformation should be derived from the geometry and is to be interpreted as recession velocity of the galaxies.

The models of the de Sitter family have pressure and mass density, if one re-interprets the cosmological constant. However, both variables are independent of time. This is a property that is not quite apparent from the point of view of an expanding cosmological model. For the dS model Lemaître has found a coordinate transformation, which allows us to pass from the non-comoving coordinate system to the coordinate system which follows the expansion of the universe. Florides [3], Mitra [4], and Melia [5,6] have provided coordinate transformations for the de Sitter cosmos, Lanczos cosmos, the Lanczos-like cosmos, and the anti-de Sitter cosmos. These coordinate transformations have been assigned by us [1,2] to pseudo-rotations or Lorentz transformations, which establish the connection between comoving and non-comoving observers. But we have assumed that the existence of such a transformation does not necessarily infer an expansion of these models. These models have in their static forms physically less explainable forces acting from any point of the universe into all directions. The comoving systems follow these forces. Families of observers move away from these points in 'free fall'. Similar to Einstein's elevator the above-mentioned forces do not occur any longer in the moving system. One can assume by all means that all four models are static provided one defines freely falling reference systems to explain the motion of the observers. Constant pressure, constant mass density, the invariance of the stress-energy tensor ([2], Mitra [7]), and the constant spatial curvature of the universe suggest this interpretation. All four models are not well suited to describe a universe with pressure.

Melia [6,7] and Mitra [8] have tried to assign non-comoving coordinates to the Friedman model, but they could not specify a complete transformation. We suppose that there do not exist such coordinates and therefore we will not apply the coordinate technology to our model. As seed metric for our model with pressure we choose the metric of the positively curved and therefore closed de Sitter model. We extend it to a genuine expanding model by making the pseudo-hyper sphere, which forms the basic geometric structure of the model, time-dependent. We therefore demand that

$$\mathcal{R} = \mathcal{R}(t) \tag{3.1}$$

Therefore we do not look for an 'expanding metric'. Should there be an expanding metric, then this metric would not only describe the curvature properties of space, but also its expansion. We believe that the possibility is not large enough for such a geometric constuct to exist. Neither is there an expanding coordinate system. Only the coordinate system of the dS cosmos and its metric remain. This coordinate system is sufficient to perform all the necessary mathematical operations. The expanding universe consists of a series of similar dS cosms. At any time, the geometry of the pressure cosmos is a snapshot of the dS-cosmos.

Instead of coordinate systems we consider observer systems that are defined in each point of space. They consist of four orthogonal unit vectors, one of them is time-like, and the others are space-like. There are two kinds of these tetrads. Those which follow the expansion of the cosmos, and those which do not join the motion. Between the two systems a Lorentz transformation acts. The velocity parameter of the Lorentz transformation corresponds to the recession velocity of the galaxies. The fact that we cannot use the coordinate systems for the expanding universe, is not a loss. The geometry of the universe and its physical parameters can be fully described by the tetrad method. The mathematical basis for it was provided by the Italian mathematicians Ricci, Bianchi and Levi-Cività about 1900 and was briefly described in Sec. 2 on the basis of the dS-cosmos.

We start from the structure of the dS-cosmos with the line element

$$ds^2 = \frac{1}{\cos^2\eta}dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2 - \cos^2\eta dt^2 . \tag{3.2}$$

Therein is

$$a = 1/\alpha = \cos\eta = \sqrt{1 - r^2/\mathbb{R}^2} \tag{3.3}$$

The radius $\mathbb{R}$ of the pseudo-hyper sphere is time-dependent in accordance with (3.1). The relative velocity between the comoving and non-comoving observer systems we take from the dS model

$$v = \frac{r}{\mathbb{R}} = \sin\eta \tag{3.4}$$

and we note that this velocity is closely connected with the geometry, namely with the polar angle $\eta$.

The lateral field quantities of (2.8) transform homogeneously into the comoving observer system

$$B_{m'} = L^m_{m'} B_m, \quad C_{m'} = L^m_{m'} C_m . \tag{3.5}$$

One obtains

$$\begin{aligned} B_{m'} &= \left\{\alpha a \frac{1}{r}, 0, 0, -i\alpha v a \frac{1}{r}\right\} = \left\{\frac{1}{r}, 0, 0, -\frac{i}{\mathbb{R}}\right\} \\ C_{m'} &= \left\{\alpha a \frac{1}{r}, \frac{1}{r}\cot\vartheta, 0, -i\alpha v a \frac{1}{r}\right\} = \left\{\frac{1}{r}, \frac{1}{r}\cot\vartheta, 0, -\frac{i}{\mathbb{R}}\right\} \end{aligned}. \tag{3.6}$$

Due to the structure of the first three components of the second parenthesized expressions one might assume that a curved geometry could be made flat by a Lorentz transformation. However, the relation

$$\alpha a = 1. \tag{3.7}$$

is valid. Therein $\alpha$ is the Lorentz factor, ie a kinematic quantity. By contrast, a is a geometrical variable which is related to the curvature of space. However, since the relative velocity is linked to the structure of the space, also the quantities $\alpha$ and a are connected by (3.7) and thus simplify the components of the lateral field quantities. However, they still contain information about the curvature of space.

The time-like components of the quantities B and C in (3.6) are equal. We can correctly surmise that the fourth component of the radial field quantity U of (2.7) accepts the same value in the comoving system, so that the expansion scalar has the simple form

$$'u^{s'}{}_{||s'} = \left(B_{s'} + C_{s'} + 'U_{s'}\right)'u^{s'} = -3\frac{i}{\mathcal{R}}, \quad 'u^{s'} = \{0,0,0,1\}. \tag{3.8}$$

Moreover, we assume that the universe is expanding in free fall. In accordance with the principle of Einstein's elevator no radial forces occur in the comoving system. Thus, we have $'U_{1'} = 0$ and have exploited the third and last field quantity of the system

$$'U_{m'} = \left\{0,0,0,-\frac{i}{\mathcal{R}}\right\}. \tag{3.9}$$

However, at the beginning we have demanded that the radius of curvature of the universe is time-dependent, but position-independent. Therefore, we define

$$\mathcal{F}_{m'} = \frac{1}{\mathcal{R}}\mathcal{R}_{|m'} = \left\{0,0,0,\frac{1}{\mathcal{R}}\mathcal{R}_{|4'}\right\}. \tag{3.10}$$

It will be seen that

$$\frac{1}{\mathcal{R}}\mathcal{R}_{|4'} = -\frac{i}{\mathcal{R}} \tag{3.11}$$

is a viable approach. Thus, the numerical identities

$$\mathcal{F}_{4'} \stackrel{*}{=} B_{4'} \stackrel{*}{=} C_{4'} \stackrel{*}{=} 'U_{4'} \ . \tag{3.12}$$

apply. From (3.9) the relation

$$'U^{s'}{}_{\underset{1}{||}s'} + 'U^{s'}\,'U_{s'} = 0 \tag{3.13}$$

immediately follows. With $\partial_{4'} = \partial / i\partial T'$ and the proper time $T'$ of the comoving observer

$$\frac{1}{\mathcal{R}}\mathcal{R}^{\bullet} - \frac{1}{\mathcal{R}} = 0, \quad \mathcal{R}^{\bullet} = 1, \quad \mathcal{R}^{\bullet\bullet} = 0 \tag{3.14}$$

corresponds to the Friedman equation. For the lateral field quantities the relations

$$\begin{aligned} &B_{m'\underset{2}{||}n'} + B_{m'}B_{n'} = -'m_{m'}\,'m_{n'}\frac{1}{\mathcal{R}^2}, \qquad B^{s'}{}_{\underset{2}{||}s'} + B^{s'}B_{s'} = -\frac{1}{\mathcal{R}^2} \\ &C_{m'\underset{3}{||}n'} + C_{m'}C_{n'} = -\left('m_{m'}\,'m_{n'} + b_{m'}b_{n'}\right)\frac{1}{\mathcal{R}^2}, \qquad C^{s'}{}_{\underset{3}{||}s'} + C^{s'}C_{s'} = -\frac{2}{\mathcal{R}^2} \end{aligned} \tag{3.15}$$

arise. Therein the graded derivatives according to (2.9) have been used, this time in the primed form. One obtains the stress-energy tensor

$$\begin{aligned} T_{m'n'} &= -pg_{m'n'} + \left(p + \mu_0\right)'u_{m'}\,'u_{n'} \\ \kappa p &= -\frac{1}{\mathcal{R}^2}, \quad \kappa\mu_0 = \frac{3}{\mathcal{R}^2} \end{aligned} \ . \tag{3.16}$$

The equation of state for the cosmos is

$$p = -\frac{1}{3}\mu_0 . \tag{3.17}$$

The conservation law

$$T^{m'n'}{}_{||n'}=0,\quad p_{|\alpha'}=0,\quad \mu_{0|4'}=-3\left(p+\mu_0\right)\mathcal{F}_{4'},\quad \alpha=1,2,3 \tag{3.18}$$

confirms that the pressure is independent of position and that with (3.10) and (3.11) the quantiy $\mathcal{F}$ has been chosen correctly. Substituting the values of p and $\mu_0$ into (3.18), second equation, one gains for $\mathcal{F}$ the anticipated term. To the above relations

$$p_{|4'}=\left(p+\mu_0\right)\mathcal{F}_{4'} \tag{3.19}$$

can still be added. $p+\mu_0$ is often called effective mass.

# 4. THE NON-COMOVING SYSTEM

If our pressure model is supposed to be useful in the creation of a cosmological model which describes Nature reasonably well, then we must also be able to specify all quantities and relations of the model in the non-comoving system. We note right away at the outset that a transformation to this system does not lead to the static dS system, which we have used as a starting model. Only the lateral field quantities B and C, which transform like tensors according to (3.5), take the dS-form. The radial quantity U which essentially describes the expansion of the cosmos must be re-derived from the Ricci-rotation coefficients. However, these transform inhomogeneously from the comoving to the non-comoving system

$$A_{mn}{}^{s}=L_{m\,n\,s'}^{m'n's}\,{}'A_{m'n'}{}^{s'}+L_{mn}{}^{s},\quad L_{mn}{}^{s}=L_{s'}^{s}\,L_{n|m}^{s'}. \tag{4.1}$$

Since the Lorentz transformation is a pseudo-rotation in the [1,4]-subspace, the above relation is simplified to

$$U_{mn}{}^{s}={}'U_{mn}{}^{s}+L_{mn}{}^{s}. \tag{4.2}$$

With

$$\begin{gathered} U_{mn}{}^{s}=h_m{}^{s}U_n-h_{mn}U^{s},\\ L_{mn}{}^{s}=h_m{}^{s}L_n-h_{mn}L^{s},\quad L_n=L_{sn}{}^{s}=\left\{L_{41}{}^{4},L_{14}{}^{1}\right\} \end{gathered} \tag{4.3}$$

one finally gains the simple relation

$$U_m={}'U_m+L_m. \tag{4.4}$$

Taking into account (4.1) and (2.2), one first has

$$L_1=-i\alpha^2 v_{|4},\quad L_4=i\alpha^2 v_{|1}.$$

With the definition of the relative velocity (3.4) we obtain the auxiliary relation

$$v_{|m}=\left\{1,0,0,0\right\}\frac{a}{\mathcal{R}}-v\mathcal{F}_m,\qquad \mathcal{F}_m=L_m^{m'}\mathcal{F}_{m'}=\left\{-\alpha v\frac{1}{\mathcal{R}},0,0,-\alpha\frac{i}{\mathcal{R}}\right\} \tag{4.5}$$

and finally

$$L_m = \left\{\alpha^3 v \frac{1}{\mathcal{R}}, 0, 0, i\,\alpha^3 \frac{1}{\mathcal{R}}\right\} = \left\{-i\alpha v, 0, 0, \alpha\right\} i\alpha^2 \frac{1}{\mathcal{R}}. \tag{4.6}$$

From (4.4) one obtains

$$U_m = \left\{\alpha^3 v^3 \frac{1}{\mathcal{R}}, 0, 0, i\alpha^3 v^2 \frac{1}{\mathcal{R}}\right\}, \tag{4.7}$$

a quantity which apparantly differs from the static dS-expression. However, it may be in accordance with

$$U_m = \hat{U}_m + f_m, \quad \hat{U}_m = \left\{-\alpha v \frac{1}{\mathcal{R}}, 0, 0, 0\right\}, \quad f_m = \left\{i\alpha^2 v \mathcal{F}_4, 0, 0, -i\alpha^2 v \mathcal{F}_1\right\}$$

decomposed in such a way that after switching off the expansion $(\mathcal{F} = 0)$ only the dS-expression for the radial force [2] remains. In this context it can be discussed, wether the radial field quantity in the non-comoving system can be derived from a metric coefficient. If this is not the case, there does not exist a non-comoving coordinate system either. It is easy to find $f_m = (\ln\alpha)_{|m}$. However, the dS-piece of the quantity $\hat{U}$ is only a gradient if is $\mathcal{R} = \text{const.}$, ie if the pressure model is reduced to the dS model. We recognize that a Lorentz transformation of the reference system is not always accompanied by a transformation of the coordinate system.

Differentiating (4.7) we obtain the relation

$$U^s{}_{\underset{1}{||s}} + U^s U_s = 0\,. \tag{4.8}$$

A comparison with (3.13) shows that the U-equations are form-invariant under a Lorentz transformation. In addressing the B- and C-equations one must again consider (4.5)

$$B_{\underset{2}{m||n}} + B_m B_n = -m_m m_n \frac{1}{\mathcal{R}^2} + \begin{pmatrix} -\alpha^2 v^2 \frac{1}{\mathcal{R}^2} & & & -i\alpha^2 v \frac{1}{\mathcal{R}^2} \\ & 0 & & \\ & & 0 & \\ -i\alpha^2 v \frac{1}{\mathcal{R}^2} & & & \alpha^2 v^2 \frac{1}{\mathcal{R}^2} \end{pmatrix},$$

$$C_{\underset{3}{m||n}} + C_m C_n = -\left(m_m m_n + b_m b_n\right) \frac{1}{\mathcal{R}^2} + \begin{pmatrix} -\alpha^2 v^2 \frac{1}{\mathcal{R}^2} & & & -i\alpha^2 v \frac{1}{\mathcal{R}^2} \\ & 0 & & \\ & & 0 & \\ -i\alpha^2 v \frac{1}{\mathcal{R}^2} & & & \alpha^2 v^2 \frac{1}{\mathcal{R}^2} \end{pmatrix}, \tag{4.9}$$

$$B^s{}_{\underset{2}{||s}} + B^s B_s = -\frac{1}{\mathcal{R}^2}, \quad C^s{}_{\underset{3}{||s}} + C^s C_s = -\frac{2}{\mathcal{R}^2}.$$

For the field equations, the relations (2.10) can be used. For the stress-energy tensor, we expect

$$T_{11} = -p - \alpha^2 v^2 \left(p + \mu_0\right), \quad T_{22} = -p, \quad T_{33} = -p,$$
$$T_{41} = -i\alpha^2 v\left(p + \mu_0\right), \quad T_{44} = \mu_0 + \alpha^2 v^2 \left(p + \mu_0\right). \tag{4.10}$$

The relations (4.8) and (4.9) supply with the values from (3.16) the above expressions.

# 5. DISCUSSION OF THE MODEL

The pressure model is geometrically based on a pseudo-hypersphere with a time-dependent radius and shows some interesting features. From the 'U-equation we have exploited the relation $\mathcal{R}^{\bullet} = 1$. It follows $\mathcal{R}^{\bullet\bullet} = 0$. The expansion of the universe is constant. We borrow from (3.4)

$$r = \mathcal{R} \sin\eta \tag{5.1}$$

with the polar angle $\eta$ of the pseudo-hyper sphere. If an observer does not perform a particular motion then is $\eta = \text{const.}$. Differentiating (5.1) leads to the Hubble law

$$r^{\bullet} = \frac{1}{\mathcal{R}} \mathcal{R}^{\bullet} r = Hr\,. \tag{5.2}$$

At the equator $\left(r = \mathcal{R}\right)$ of the pseudo-hyper sphere one has $r^{\bullet} = \mathcal{R}^{\bullet} = 1$ or in physical units

$$v = r^{\bullet} = c\,, \tag{5.3}$$

wherein again the definition of the velocity (3.4) was considered. The expansion-related recession velocity of galaxies has the highest attainable value at the equator, the velocity of light. A galactic island formation is not possible in this model. The model has a horizon at

$$r_H = cT'\,. \tag{5.4}$$

No signal beyond the horizon can reach an observer situated at $\eta = 0$. Since all points on the hypersphere are equivalent, each observer has his individual horizon at an arbitrary position in the universe.

Further, we want to survey if the definition of the velocity

$$v = r^{\bullet} = \frac{dr}{dT'} \tag{5.5}$$

(with the proper time $T'$ of the comoving observer) complies with the velocity definition of the theory of relativity. An observer in the non-comoving system identifies the radial velocity of a receding galaxy as

$$v = \frac{dx^1}{dT}\,. \tag{5.6}$$

Therein T is the observer's proper time. At any time during the expansion, the radial arc element is determined by the dS metric

$$dx^1 = \alpha dr, \quad \alpha = 1/a = 1/\cos\eta = 1/\sqrt{1 - r^2/\mathcal{R}^2}\,. \tag{5.7}$$

The proper time of the observer depends on the Lorentz relation

$$\frac{dT}{dT'} = \alpha \,. \tag{5.8}$$

The universe expands in free fall, the Lorentz factor $\alpha$ and the metric factor $\alpha$ are identical according to (3.7) and (5.7), so that

$$v = \frac{\alpha dr}{\alpha dT'} = \dot{r} \tag{5.9}$$

proved to be the relative velocity of the observers and thus the recession velocity of the galaxies.

Is noteworthy that these results are identical with those which Melia [5-7,9] derived from a model that he called $R_h = ct$ model[2]. However, Melia gains his relations from a flat FRW ansatz. In contrast, our pressure model is positively curved and closed. This has the advantage that the Olbers' paradox need not be concealed either or be explained away by expansion effects. The question remains whether both models are identical. We start our considerations with the dS model which is based on a pseudo-hyper sphere, and thus positively curved and closed. But according to the FRW classification the dS model is called flat $(k = 0)$. It is quite possible that the FRW-classification of the Melia model does not make the correct statement.

# 6. CONCLUSIONS

We have elaborated the mathematical structure of a cosmological model which can be helpful in building a physically viable model. The model includes pressure and mass density. Relations with these quantities result from the exact solutions of Einstein's field equations. The cosmic horizon arises from the geometric structure of the model. The recession velocity of galaxies cannot exceed the velocity of light. There can be no galactic island formation.

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[2] Melia's expression agrees with (5.4). Melia's coordinate time t corresponds to the proper time of this system in the free-falling comoving system. This time is referred to by us as T '.

# 7. REFERENCES

[1] Burghardt R., *Transformations in de Sitter and Lanczos models I.* http://arg.or.at/Wpdf/WTrans1.pdf

[2] Burghardt R., *Transformations in de Sitter and Lanczos models II*. http://arg.or.at/Wpdf/WTrans2.pdf

[3] Florides P. S., *The Robertson-Walker metrics expressible in static* form. GRG **12**, 563, 1980

[4] Mitra A., *When can an 'Expanding Universe' look 'Static' and vice versa: A comprehensive study.* Journ Math.Phys. **D 24**, 155003, 2015

[5] Melia F., *Proper size of the visible universe in FRW metrics with constant spacetime curvature.* astro-ph.CO/1207.1332

[6] Melia F., Abdelqader M., *The cosmological spacetime.* astro-ph/09075394

[7] Melia F., *The cosmic horizon.* astro-ph/07114181

[8] Mitra A., *Friedmann-Robertson-Walker metric in curvature coordinates and its application.* Grav. Cosmol. 19, 134, 2013

[9] Melia F., Shevchuk A. S. H., *The* $R_h$*=ct universe.* Mon. Not. Roy. Astr. Soc. **419**, 2579, 2012