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TRANSFORMATIONS IN DE SITTER AND LANCZOS MODELS I.

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Abstract: We derive Lorentz transformations from comoving to non-comoving reference systems for four classical cosmological models, the de Sitter cosmos, the Lanczos cosmos, the Lanczos-like cosmos, and the anti-de Sitter cosmos. These Lorentz transformations are associated to known coordinate transformations, and we take from the Lorentz transformations the relative velocities between comoving and non-comoving observers. We discover differences between the usual velocity definitions in FWR models and our definitions.

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1. INTRODUCTION

Working on expanding cosmological models one recently has dealt with the problem of whether there exist for commonly used comoving coordinate systems also coordinate systems at rest and whether the metrics in these new coordinate systems are static (Florides [1], Mitra [2, 3] Melia [4,5], and Eriksen Grøn [6]). However, we believe that the finding of static coordinates is not sufficient to judge the plausibility of the physical static metric. One should assign to a static coordinate system a static frame of reference, and survey whether this reference system is connected to the expanding reference system via a pseudo-rotation and whether this pseudo-rotation has a physical content, ie whether it is a Lorentz transformation. This can be verified by calculating the field quantities in the static system and by confirming if these satisfy the conditions for a static model and whether Einstein's field equations are satisfied. As a final test one can make sure if the Ricci is invariant with respect to the pseudo-rotation.

In Sec. 2. the general procedures which are applied to cosmological models to be examined are briefly described. In Sec. 3. the embeddings of models into a 5-dimensional flat space are written down. Various forms of the metric are noted. In Sec 4. the transformations between the comoving and non-comoving systems are set up and the line elements of each other's systems are complemented. In Sec. 5. the Lorentz transformations which are assigned to the coordinate transformations are derived. From the Lorentz transformations the relative velocities between the comoving and non-comoving observers are calculated. It is emphasized that there is a difference between the definitions of these velocites and those commonly used in the Friedman models. In further paper the verification of the gained results is carried out with the help of Einstein's field equations.

2. PRELIMINARY REMARKS

The above-mentioned authors have found static coordinate systems for some expanding cosmological models. To verify the reliability of these coordinate systems, we redifferentiate given transformation equations and we calculate the transformation matrix

$$\Lambda^{i}_{i'} = x^{i}{}_{|i'} \, . \tag{2.1}$$

Therein i' denotes the comoving coordinates, comoving with the expansion of the cosmos, i denotes the static coordinates. The static metric g_{ik} is in relation with the expanding metric $g_{i'k'}$ by

$$g_{i'k'} = \Lambda^{i\,k}_{i'k'} g_{ik} \, .$$

From both metrics we read the 4-beins and we calculate the matrix of pseudo-rotation

$$L^{m}_{m'} = \overset{m}{e}_{i} \Lambda^{i}_{i'} \underset{m'}{e}{}^{i'} , \tag{2.2}$$

whereby it is to be considered wether this pseudo-rotation can be used as a Lorentz transformation.

3. THE MODELS

In the above-mentioned papers [1-6] six expanding cosmological models have been studied and coordinate transformations have been specified, which mediate between co-moving and static coordinate systems. For four of these we add to the coordinate transformation a transformation of the reference systems and we judge if this transformation is a Lorentz transformation. Furthermore, the field quantities obtained with it should be physically and geometrically plausible and should satisfy the Einstein field equations. The models examined are the de Sitter universe [9-12], the Lanczos cosmos [13], a model which is similar to the Lanczos cosmos and the anti-de Sitter model.

All four models can be explained geometrically if one takes into consideration their embeddings into a 5-dimensional flat space. The x^a are the Cartesian coordinates in the embedding space and $\eta, \vartheta, \varphi, i\psi$ pseudo-spherical coordinates.

I. For the *de Sitter-Model* one has

$$\begin{aligned} x^3 &= \mathcal{R}_0 \sin\eta \sin\vartheta \sin\varphi \\ x^2 &= \mathcal{R}_0 \sin\eta \sin\vartheta \cos\varphi \\ x^1 &= \mathcal{R}_0 \sin\eta \cos\vartheta \\ x^4 &= \mathcal{R}_0 \cos\eta \sin i\psi \\ x^0 &= \mathcal{R}_0 \cos\eta \cos i\psi \end{aligned} \qquad (3.1)$$

and

$$x^a x^a = \mathcal{R}_0^2, \quad a = 0,1,\ldots,4, \quad \mathcal{R}_0 = \text{const.} \qquad (3.2)$$

is the equation of a pseudo-hyper sphere with the (also time-independent) radius $\mathcal{R}_0$. The de Sitter universe has positive curvature and is closed.

$$\text{(I B)} \qquad ds^2 = \mathcal{R}_0^2 d\eta^2 + \mathcal{R}_0^2 \sin^2\eta\, d\vartheta^2 + \mathcal{R}_0^2 \sin^2\eta \sin^2\vartheta\, d\varphi^2 + \mathcal{R}_0^2 \cos^2\eta\, di\psi^2 \qquad (3.3)$$

is the line element on this pseudo-hyper sphere which also can be written with

$$r = \mathcal{R}_0 \sin\eta \qquad (3.4)$$

as

$$\text{(I B')} \qquad ds^2 = \frac{1}{\cos^2\eta} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 - \cos^2\eta\, dt^2 \qquad (3.5)$$

or

$$\text{(I B'')} \qquad ds^2 = \frac{1}{1 - \frac{r^2}{\mathcal{R}_0^2}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 - \left(1 - \frac{r^2}{\mathcal{R}_0^2}\right) dt^2 \ . \qquad (3.6)$$

The radius of the sphere in the 3-dimensional spherical space is $r^2 = x^\alpha x^\alpha,\ \alpha = 1,2,3$ and

$$\mathcal{R}_0 d i\psi = i dt \qquad (3.7)$$

the coordinate-time interval. We remember that de Sitter has described by (3.5) a static cosmos. The expanding version was added later. We note that the de Sitter model is based on a time-independent geometric framework. The coordinate system used here is the *non-comoving* one.

II. The *model of Lanczos* has the embedding

$$\begin{aligned} x^3 &= \mathcal{R}_0 \cos i\psi \sin\eta \sin\vartheta \sin\varphi \\ x^2 &= \mathcal{R}_0 \cos i\psi \sin\eta \sin\vartheta \cos\varphi \\ x^1 &= \mathcal{R}_0 \cos i\psi \sin\eta \cos\vartheta \quad , \\ x^0 &= \mathcal{R}_0 \cos i\psi \cos\eta \\ x^4 &= \mathcal{R}_0 \sin i\psi \end{aligned} \qquad (3.8)$$

wherein $x^a x^a = \mathcal{R}_0^2, \quad a = 0,1,...,4, \quad \mathcal{R}_0 = \text{const.}$ again describes a 4-dimensional sphere in a 5-dimensional embedding space with the time-independent radius $\mathcal{R}_0$. The 3-dimensional spherical surface

$$x^\mu x^\mu = \mathcal{R}^2, \quad \mu = 0,1,2,3, \quad \mathcal{R} = \mathcal{R}_0 \cos i\psi = \mathcal{R}_0 \mathrm{ch}\psi \qquad (3.9)$$

with the time-dependent radius $\mathcal{R} = \mathcal{R}(t)$ describes the 3-dimensional space of our experience and can be considered as expanding/contracting. The Lanczos cosmos is positively curved and closed. The line element has the form

(II A) $$ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \sin^2\eta d\vartheta^2 + \mathcal{R}^2 \sin^2\eta \sin^2\vartheta d\varphi^2 + \mathcal{R}_0^2 \, di\psi^2 . \qquad (3.10)$$

Therein is

$$\mathcal{R} = \mathcal{K}\mathcal{R}_0, \quad \mathcal{K} = \cos i\psi = \mathrm{ch}\psi , \qquad (3.11)$$

$\mathcal{K}$ the scale factor, and $dt = \mathcal{R}_0 d\psi$ the cosmic time valid for all observers. With

$$r = \mathcal{R}_0 \sin\eta \qquad (3.12)$$

the metric can also be written as

(II A') $$ds^2 = \mathcal{K}^2 \left[\frac{1}{\cos^2\eta} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 \right] - dt^2 , \qquad (3.13)$$

(II A'') $$ds^2 = \mathcal{K}^2 \left[\frac{1}{1 - \frac{r^2}{\mathcal{R}_0}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 \right] - dt^2 . \qquad (3.14)$$

We note that in this approach the coordinate system is the *comoving* coordinate system, and the non-comoving must be searched for.

III. For the *Lanczos-like* model one has the embedding

$$\begin{aligned} x^3 &= \mathcal{R}_0 \mathrm{sh}\psi\, \mathrm{sh}\eta \sin\vartheta \sin\varphi \\ x^2 &= \mathcal{R}_0 \mathrm{sh}\psi\, \mathrm{sh}\eta \sin\vartheta \cos\varphi \\ x^1 &= \mathcal{R}_0 \mathrm{sh}\psi\, \mathrm{sh}\eta \cos\vartheta \\ x^4 &= i\mathcal{R}_0 \mathrm{sh}\psi\, \mathrm{ch}\eta \\ x^0 &= \mathcal{R}_0 \mathrm{ch}\psi \end{aligned} \quad . \tag{3.15}$$

Again

$$x^a x^a = \mathcal{R}_0^2, \quad x^m x^m = \left(i\mathcal{R}\right)^2, \quad m = 1,2,...4, \quad \mathcal{R}\left(t\right) = \mathcal{R}_0 \mathrm{sh}\psi \tag{3.16}$$

applies to the spherical surfaces and

$$\text{(III A)} \qquad ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2\, \mathrm{sh}^2\eta d\vartheta^2 + \mathcal{R}^2\, \mathrm{sh}^2\eta \sin^2\vartheta d\varphi^2 + \mathcal{R}_0^2\, di\psi^2 \tag{3.17}$$

to the metric. With

$$\mathcal{R} = \mathcal{K}\mathcal{R}_0, \quad \mathcal{K} = \mathrm{sh}\psi, \quad r = \mathcal{R}_0\, \mathrm{sh}\eta \tag{3.18}$$

one obtains from (3.17)

$$\text{(III A')} \qquad ds^2 = \mathcal{K}^2 \left[\frac{1}{\mathrm{ch}^2\eta} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 \right] - dt^2, \tag{3.19}$$

$$\text{(III A'')} \qquad ds^2 = \mathcal{K}^2 \left[\frac{1}{1 + \frac{r^2}{\mathcal{R}_0}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 \right] - dt^2. \tag{3.20}$$

The Lanczos-like model expands, is negatively curved, and infinite.

IV. The *anti-de Sitter model* has the embedding

$$\begin{aligned} x^3 &= \mathcal{R}_0 \mathrm{sh}\eta \sin\vartheta \sin\varphi \\ x^2 &= \mathcal{R}_0 \mathrm{sh}\eta \sin\vartheta \cos\varphi \\ x^1 &= \mathcal{R}_0\, \mathrm{sh}\eta \cos\vartheta \\ x^4 &= i\mathcal{R}_0\, \mathrm{ch}\eta \sin\psi \\ x^0 &= i\mathcal{R}_0 \mathrm{ch}\eta \cos\psi \end{aligned} \quad . \tag{3.21}$$

It applies

$$x^a x^a = \left(i\mathcal{R}_0\right)^2, \quad x^\alpha x^\alpha = \mathcal{R}_0^2\, \mathrm{sh}^2\eta, \quad \alpha = 1,2,3, \quad x^4 x^4 + x^0 x^0 = \left(i\mathcal{R}_0 \mathrm{ch}\eta\right)^2. \tag{3.22}$$

The AdS universe is spatially infinite and periodic in the time $dt = \mathcal{R}_0 d\psi$. With

$$r = \mathcal{R}_0 \mathrm{sh}\eta \tag{3.23}$$

the line elements can be written as

(IV B) $$ds^2 = \mathcal{R}_0^2 d\eta^2 + \mathcal{R}_0^2 \,\mathrm{sh}^2\eta\, d\vartheta^2 + \mathcal{R}_0^2 \,\mathrm{sh}^2\eta \sin^2\vartheta\, d\varphi^2 + \mathcal{R}_0^2 \,\mathrm{ch}^2\eta\, di\psi^2 , \qquad (3.24)$$

(IV B') $$ds^2 = \frac{1}{\mathrm{ch}^2\eta} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 - \mathrm{ch}^2\eta\, dt^2 , \qquad (3.25)$$

(IV B'') $$ds^2 = \frac{1}{1+\frac{r^2}{\mathcal{R}_0^2}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 - \left(1+\frac{r^2}{\mathcal{R}_0^2}\right) dt^2 \qquad (3.26)$$

and in the same way as in the de Sitter model they are given in non-comoving coordinate systems.

We have envisaged four cosmological models, all based on a fixed geometric structure, a pseudo-hyper sphere with time-constant radius. The pseudo-hyper sphere is embedded in a 5-dimensional flat complex space, and is interpreted as a one-shell hyperboloid in the pseudo-real representation. The latter representation has the advantage that the (temporal) infinity of the pseudo-hypersphere can be shown graphically. The radius of the pseudo-hyper sphere then appears as the radius of the circle at the waist of the hyperboloid, but the curvature of the radial hyperbolae decreases outwards. However, the pseudo-hyper sphere has the same curvature everywhere, even at infinity. The fact that its pseudo-real representation results in a hyperboloid a pseudo-hyper sphere is also called a *hyperboloid of constant curvature*.

4. THE TRANSFORMATIONS

In the following the coordinate transformations given by Florides and Mitra will be applied to the aforementioned cosmological models. First, the matrix of the coordinate transformation, and thus the metric of the relatively moving or the static system is calculated according to (2.1). From this metric the tetrads are extracted and the Lorentz transformation is calculated with (2.2). From this, in turn, we extract the velocity parameter and thus obtain the recession velocities of the galaxies.

I. The model by *de Sitter* was originally designed as a static one with the metric (3.5). This metric was brought by Lemaître [14,15] with a coordinate transformation into a form that suggests assuming the cosmos to be expanding. Before we deal with this transformation, we investigate the metric (3.5) in more detail.

From

$$dx^1 = \frac{1}{\cos\eta} dr, \quad dx^4 = \cos\eta\, idt \qquad (4.1)$$

we read the 4-bein system

$$\overset{1}{e}_1 = \frac{1}{\cos\eta}, \quad \overset{4}{e}_4 = \cos\eta \qquad (4.2)$$

and from the Ricci-rotation coefficients of the geometry we obtain

$$E_1 = -A_{41}{}^1 = \overset{4}{e}_4 \overset{4}{e}_{4|1} = -\frac{1}{\cos\eta} (\cos\eta)_{|1} = \tan\eta\, \eta_{|1}$$

and from (3.4) $\eta_{|1} = 1/\mathcal{R}_0$. Thus, a static force

$$E_1 = \frac{1}{\mathcal{R}_0}\tan\eta \tag{4.3}$$

occurs in any point of the de Sitter model, pointing into all directions, and pulling away all space points from the observer. Therefore, we want the metric in the forms (3.3) - (3.6) not to be called static, but expanding in non-comoving coordinates.

Between the comoving system (r',t') and the non-comoving system (r,t) the coordinate transformation

$$r = \mathcal{K}r', \quad \mathcal{K} = e^{\psi'}, \quad \psi' = \psi + \ln\cos\eta, \quad e^{\psi'} = e^{\psi}\cos\eta \tag{4.4}$$

mediates. Therein r' is the radial coordinate comoving with the expansion, $t' = \mathcal{R}_0\psi'$ the cosmic time, and $\mathcal{K}$ the scale factor depending only on the time. We immediately have

$$dr = \mathcal{K}dr' + r\frac{1}{\mathcal{K}}\mathcal{K}_{|4'}dx^{4'}, \quad dx^{4'} = i\mathcal{R}_0 d\psi'.$$

With $\partial_{4'} = \partial / i\mathcal{R}_0\partial\psi'$ and (4.4) we find

$$\frac{1}{\mathcal{K}}\mathcal{K}_{|m'} = \left\{0,0,0,-\frac{i}{\mathcal{R}_0}\right\}. \tag{4.5}$$

The second equation (4.4) we write with $\cos^2\eta = 1 - r^2/\mathcal{R}_0^2, \quad r = e^{\psi'}r'$ as

$$e^{-2\psi} = e^{-2\psi'} - \frac{r'^2}{\mathcal{R}_0^2}\ .$$

Differentiation yields

$$d\psi = e^{2(\psi-\psi')}d\psi' + e^{2\psi}\frac{r'}{\mathcal{R}_0^2}dr'\ .$$

If we use (3.4) and put

$$\alpha = \frac{1}{\cos\eta} = \frac{1}{\sqrt{1 - r^2/\mathcal{R}_0^2}}, \quad v = \frac{r}{\mathcal{R}_0}, \quad \alpha^2 = 1 + \alpha^2 v^2 \tag{4.6}$$

we have adapted the notation for later use. Thus, we get the matrices for the coordinate transformation

$$\Lambda^{i}_{i'} = \begin{pmatrix} \mathcal{K} & & & -iv \\ & 1 & & \\ & & 1 & \\ i\mathcal{K}\alpha^2 v & & & \alpha^2 \end{pmatrix}, \quad \Lambda^{i'}_{i} = \begin{pmatrix} \frac{\alpha^2}{\mathcal{K}} & & & \frac{iv}{\mathcal{K}} \\ & 1 & & \\ & & 1 & \\ -i\alpha^2 v & & & 1 \end{pmatrix} \tag{4.7}$$

and thus we are able to transform the metric (3.5) into the expanding form:

(I A') $$ds^2 = e^{2\psi'}\left[dr'^2 + r'^2 d\vartheta^2 + r'^2\sin^2\vartheta d\varphi^2\right] - dt'^2\ . \tag{4.8}$$

The de Sitter universe is expanding in free fall. For a comoving observer the universe appears to be flat. We will discuss it later.

II. With the *model of Lanczos* the reverse order is necessary. The line element (3.10) of the model is presented in the comoving coordinate system. In order to adapt the notation to the previous problem, we have to rewrite the line element in the primed form

$$\text{(II A')} \qquad ds^2 = \mathcal{K}^2\left[\frac{1}{\cos^2\eta'}dr'^2 + r'^2 d\vartheta^2 + r'^2\sin^2\vartheta d\varphi^2\right] - dt'^2 \tag{4.9}$$

using

$$r' = \mathcal{R}_0 \sin\eta', \quad dt' = \mathcal{R}_0 d\psi', \quad \mathcal{R} = \mathcal{K}\mathcal{R}_0, \quad \mathcal{K} = \mathrm{ch}\psi'. \tag{4.10}$$

For the non-comoving System we expect

$$r = \mathcal{R}_0 \sin\eta, \quad dt = \mathcal{R}_0 d\psi, \quad r = \mathcal{K} r'. \tag{4.11}$$

By comparing (4.10) and (4.11) we obtain the relation of the polar angles

$$\sin\eta = \sin\eta' \mathrm{ch}\psi' \tag{4.12}$$

and we write down the two auxiliary relations for later use

$$\cos^2\eta = \cos^2\eta' - \sin^2\eta' \mathrm{sh}^2\psi', \quad \cos^2\eta = \cos^2\eta' \mathrm{ch}^2\psi' - \mathrm{sh}^2\psi'. \tag{4.13}$$

We obtain the transformation matrix for the non-comoving coordinates with

$$r = \mathcal{K} r', \quad \mathrm{th}\psi = \frac{1}{\cos\eta'}\mathrm{th}\psi'. \tag{4.14}$$

After differentiation and the use of the auxiliary relations (4.13) one has

$$\Lambda^{i}_{i'} = \begin{pmatrix} \mathrm{ch}\psi' & & & -i\sin\eta'\mathrm{sh}\psi' \\ & 1 & & \\ & & 1 & \\ i\dfrac{\tan\eta}{\cos\eta}\dfrac{\mathrm{sh}\psi'}{\cos\eta'} & & & \dfrac{\cos\eta'}{\cos^2\eta} \end{pmatrix}$$

$$\Lambda^{i'}_{i} = \begin{pmatrix} \dfrac{\cos^2\eta'}{\cos^2\eta}\dfrac{1}{\mathrm{ch}\psi'} & & & i\sin\eta'\cos\eta'\mathrm{th}\psi' \\ & 1 & & \\ & & 1 & \\ -i\dfrac{\tan\eta}{\cos\eta}\mathrm{th}\psi' & & & \cos\eta' \end{pmatrix}. \tag{4.15}$$

This yields the line elements in the forms

$$\text{(II B)} \qquad ds^2 = \mathcal{R}_0^2 d\eta^2 + \mathcal{R}_0^2\sin^2\eta d\vartheta^2 + \mathcal{R}_0^2\sin^2\eta\sin^2\vartheta d\varphi^2 + \mathcal{R}_0^2\cos^2\eta di\psi^2, \tag{4.16}$$

$$\text{(II B')} \qquad ds^2 = \frac{1}{\cos^2\eta}dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2 - \cos^2\eta dt^2, \tag{4.17}$$

(II B")
$$ds^2 = \frac{1}{1-\frac{r^2}{\mathcal{R}_0^2}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \left(1-\frac{r^2}{\mathcal{R}_0^2}\right) dt^2 . \tag{4.18}$$

As with the de Sitter model, the line element relates to a pseudo-hyper sphere.

III. The *Lanczos-like* model is based on the metric (3.20). We write this metric now as

(III A")
$$ds^2 = \mathcal{K}^2 \left[\frac{1}{1+\frac{r'^2}{\mathcal{R}_0^2}} dr'^2 + r'^2 d\vartheta^2 + r'^2 \sin^2\vartheta \, d\varphi^2 \right] - dt'^2 . \tag{4.19}$$

The following relations hold

$$r' = \mathcal{R}_0 \, \mathrm{sh}\eta', \quad dt' = \mathcal{R}_0 d\psi', \quad \mathcal{R} = \mathcal{K}\mathcal{R}_0, \quad \mathcal{K} = \mathrm{sh}\psi' \tag{4.20}$$

and for the non-comoving system

$$r = \mathcal{R}_0 \, \mathrm{sh}\eta, \quad dt = \mathcal{R}_0 d\psi, \quad r = \mathcal{K} r' . \tag{4.21}$$

By comparing the last two relations we get

$$\mathrm{sh}\eta = \mathrm{sh}\eta' \, \mathrm{sh}\psi' \tag{4.22}$$

and from this the two auxiliary relations

$$\mathrm{ch}^2\psi' - \mathrm{ch}^2\eta' \mathrm{sh}^2\psi' = 1 - \mathrm{sh}^2\eta, \quad \mathrm{ch}^2\eta' - \mathrm{sh}^2\eta' \mathrm{ch}^2\psi' = 1 - \mathrm{sh}^2\eta . \tag{4.23}$$

For calculation of the transformation matrix we use

$$r = \mathcal{R} r', \quad \mathrm{th}\psi = \mathrm{ch}\eta' \, \mathrm{th}\psi' \tag{4.24}$$

and we get

$$\Lambda^i_{i'} = \begin{pmatrix} \mathrm{sh}\psi' & & & -i\,\mathrm{sh}\eta' \mathrm{ch}\psi' \\ & 1 & & \\ & & 1 & \\ \frac{i}{1-\mathrm{sh}^2\eta} \mathrm{th}\eta' \mathrm{sh}\psi' \mathrm{ch}\psi' & & & \frac{1}{1-\mathrm{sh}^2\eta} \mathrm{ch}\eta' \end{pmatrix}$$

$$\Lambda^{i'}_i = \begin{pmatrix} \frac{1}{1-\mathrm{sh}^2\eta} \frac{\mathrm{ch}^2\eta'}{\mathrm{sh}\psi'} & & & i\,\mathrm{sh}\eta' \mathrm{ch}\eta' \mathrm{cth}\psi' \\ & 1 & & \\ & & 1 & \\ -i \frac{1}{1-\mathrm{sh}^2\eta} \mathrm{sh}\eta' \mathrm{ch}\psi' & & & \mathrm{ch}\eta' \end{pmatrix} . \tag{4.25}$$

Thus, transforming to the non-comoving metric the line element takes the forms

$$\text{(III B')} \qquad ds^2 = \frac{1}{1-\text{sh}^2\eta} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \left(1-\text{sh}^2\eta\right) dt^2, \qquad (4.26)$$

$$\text{(III B'')} \qquad ds^2 = \frac{1}{1-\frac{r^2}{\mathcal{R}_0^2}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \left(1-\frac{r^2}{\mathcal{R}_0^2}\right) dt^2. \qquad (4.27)$$

IV. The line element of the *anti-de Sitter model* in the comoving coordinate system is reported in the literature as

$$\text{(IV A)} \qquad ds^2 = \mathcal{R}^2 d\eta'^2 + \mathcal{R}^2 \text{sh}^2\eta' d\vartheta^2 + \mathcal{R}^2 \text{sh}^2\eta' \sin^2\vartheta d\varphi^2 - dt'^2. \qquad (4.28)$$

With

$$r' = \mathcal{R}_0 \text{sh}\eta', \quad \mathcal{R} = \mathcal{K}\mathcal{R}_0, \quad \mathcal{K} = \sin\psi' \qquad (4.29)$$

it can be written as

$$\text{(IV A')} \qquad ds^2 = \mathcal{K}^2 \left[\frac{1}{\text{ch}\eta'^2} dr'^2 + r'^2 d\vartheta^2 + r'^2 \sin^2\vartheta d\varphi^2\right] - dt'^2, \qquad (4.30)$$

$$\text{(IV A'')} \qquad ds^2 = \mathcal{K}^2 \left[\frac{1}{1+\frac{r'^2}{\mathcal{R}_0^2}} dr'^2 + r'^2 d\vartheta^2 + r'^2 \sin^2\vartheta d\varphi^2\right] - dt'^2. \qquad (4.31)$$

Therefore, we need only to show that the metrics of types A and B mutually transform by a suitable coordinate transformation.

For the non-comoving System we expect

$$r = \mathcal{R}_0 \text{sh}\eta, \quad r = \mathcal{K} r' \qquad (4.32)$$

and by comparing this with (4.29) we get

$$\text{sh}\eta = \text{sh}\eta' \sin\psi' \qquad (4.33)$$

and the auxiliary relations

$$\text{ch}^2\eta = \cos^2\psi' + \text{ch}^2\eta' \sin^2\psi', \quad \text{ch}^2\eta = \text{ch}^2\eta' - \text{sh}^2\eta' \cos^2\psi'. \qquad (4.34)$$

Deriving the coordinate transformation we start with

$$r = \mathcal{K} r', \quad \tan\psi = \text{ch}\eta' \tan\psi'. \qquad (4.35)$$

and we obtain the matrices

$$
\Lambda^{i}_{i'} = \begin{pmatrix} \sin\psi' & & & -i\,\mathrm{sh}\eta'\cos\psi' \\ & 1 & & \\ & & 1 & \\ \dfrac{i}{\mathrm{ch}^2\eta}\mathrm{th}\eta'\sin\psi'\cos\psi' & & & \dfrac{\mathrm{ch}\eta'}{\mathrm{ch}^2\eta} \end{pmatrix}
$$

$$
\Lambda^{i'}_{i'} = \begin{pmatrix} \dfrac{\mathrm{ch}^2\eta'}{\mathrm{ch}^2\eta\sin\psi'} & & & i\,\mathrm{sh}\eta'\mathrm{ch}\eta'\cot\psi' \\ & 1 & & \\ & & 1 & \\ -i\dfrac{\mathrm{sh}\eta'}{\mathrm{ch}^2\eta}\cos\psi' & & & \mathrm{ch}\eta' \end{pmatrix} \quad . \qquad (4.36)
$$

The coordinate transformations derived in this Section are useful, but only a precursor to the Lorentz transformations.

5. LORENTZ TRANSFORMATIONS AND VELOCITIES

Since the metric exists for all models, both in comoving and in non-comoving coordinates, we can read from the two systems the 4-bein vectors and according to (2.2) we can calculate the Lorentz transformations. From this, in turn, can be read the relative velocities between the comoving observers and observers at rest.

I. To the *de Sitter model* applies

$$
\overset{1'}{e}_{1'} = \mathcal{K}, \quad \overset{4'}{e}_{4'} = 1, \quad \overset{1}{e}_{1} = \alpha, \quad \overset{4}{e}_{4} = 1/\alpha, \quad \mathcal{K} = e^{\psi'}, \quad \alpha = 1\Big/\sqrt{1 - r^2/\mathcal{R}_0^2} \qquad (5.1)
$$

and thus the Lorentz transformation reads as

$$
L^{m}_{m'} = \begin{pmatrix} \alpha & & & -i\alpha v \\ & 1 & & \\ & & 1 & \\ i\alpha v & & & \alpha \end{pmatrix} . \qquad (5.2)
$$

The relative velocity of the observer is

$$
v = \frac{r}{\mathcal{R}_0} \qquad (5.3)
$$

and α the associated Lorentz factor. On the equator of the hypersphere ($r = \mathcal{R}_0$) v takes the value 1 (the velocity of light) , the highest posible value in the de Sitter universe. The proper time of the comoving observer coincides with the coordinate time ($dT' = dt'$). However, for the non-comoving observer is valid $dT = 1/\alpha \cdot dt$.

II. For the *Lanczos model* we obtain from the line elements of the forms of (4.9) and (4.17) the 4-bein systems

$$\overset{1'}{e}_{1'} = \frac{\mathrm{ch}\psi'}{\cos\eta'}, \quad \overset{4'}{e}_{4'} = 1; \quad \overset{1}{e}_{1} = \frac{1}{\cos\eta}, \quad \overset{4}{e}_{4} = \cos\eta \tag{5.4}$$

and from (4.15) the Lorentz transformation

$$L^{m}_{m'} = \begin{pmatrix} \frac{\cos\eta'}{\cos\eta} & & & -i\tan\eta\,\mathrm{th}\psi' \\ & 1 & & \\ & & 1 & \\ i\tan\eta\,\mathrm{th}\psi' & & & \frac{\cos\eta'}{\cos\eta} \end{pmatrix}. \tag{5.5}$$

Using (4.14), second equation, we borrow from (5.5) the relative velocity of the observers

$$v = \sin\eta\,\mathrm{th}\psi = \frac{r}{\mathcal{R}_0}\mathrm{th}\frac{t}{\mathcal{R}_0}. \tag{5.6}$$

The relative speed increases after an infinitely long time until it reaches the value of the velocity of light.

III. For the *Lanczos-like model*, we note

$$\overset{1'}{e}_{1'} = \frac{\mathrm{sh}\psi'}{\mathrm{ch}\eta'}, \quad \overset{4'}{e}_{4'} = 1; \quad \overset{1}{e}_{1} = \frac{1}{\sqrt{1-\mathrm{sh}^2\eta}}, \quad \overset{4}{e}_{4} = \sqrt{1-\mathrm{sh}^2\eta} \tag{5.7}$$

and with (4.25), the Lorentz transformation

$$L^{m}_{m'} = \frac{1}{\sqrt{1-\mathrm{sh}^2\eta}} \begin{pmatrix} \mathrm{ch}\eta' & & & -i\,\mathrm{sh}\eta'\,\mathrm{ch}\psi' \\ & 1 & & \\ & & 1 & \\ i\,\mathrm{sh}\eta'\,\mathrm{ch}\psi' & & & \mathrm{ch}\eta' \end{pmatrix}. \tag{5.8}$$

This implies the relative velocity

$$v = \mathrm{th}\eta'\,\mathrm{ch}\psi' = \mathrm{th}\eta'\,\mathrm{ch}\frac{t'}{\mathcal{R}_0}. \tag{5.9}$$

The relative speed increases infinitely after an infinitely long time.

IV. For the anti-de Sitter model one gathers from the metrics (4.30) and (3.25) the tetrads

$$\overset{1'}{e}_{1'} = \frac{\sin\psi'}{\mathrm{ch}\eta'}, \quad \overset{4'}{e}_{4'} = 1; \quad \overset{1}{e}_{1} = \frac{1}{\mathrm{ch}\eta}, \quad \overset{4}{e}_{4} = \mathrm{ch}\eta \tag{5.10}$$

and from (4.36) with the help of (2.2) one establishes the pseudo-rotation

$$L^{m}_{m'}=\begin{pmatrix} \frac{ch\eta'}{ch\eta} & & & -i\frac{th\eta}{\tan\psi'} \\ & 1 & & \\ & & 1 & \\ i\frac{th\eta}{\tan\psi'} & & & \frac{ch\eta'}{ch\eta} \end{pmatrix}. \tag{5.11}$$

From the relative speeds of the systems A and B we gather

$$v = th\eta'\cos\psi' = sh\eta\cot\psi. \tag{5.12}$$

The relative velocity is infinitely high at infinity and is periodic in time. (5.11) is not a Lorentz transformation.

6. RELATIVE VELOCITY AND RECESSION VELOCITY

We highlight again the relative velocities obtained from the Lorentz transformations and make the connection to the usual definitions of the recession velocities in the FWR models.

I. First we discuss this for the *de Sitter model*. In the comoving system one has for the radial motion $dx^{1'}/dT' = 0$, whereby the proper time dT' of an observer coincides with the coordinate time dt'. For the non-comoving observer

$$v = \frac{dx^1}{dT}, \quad \frac{dT}{dT'} = \alpha, \quad dx^1 = \alpha dr$$

applies to the relative velocity. Thus, one has

$$r^{\bullet} = \frac{dr}{dT'} = v, \quad v = \frac{r}{\mathcal{R}_0} = \sin\eta. \tag{6.1}$$

The relative velocity can only have values in the interval $[0,1]$ and cannot exceed the speed of light. One obtains the same relation from

$$r = \mathcal{R}_0\sin\eta, \quad r = \mathcal{K}r', \quad \mathcal{K} = e^{\psi'}, \quad r^{\bullet} = \frac{1}{\mathcal{K}}\mathcal{K}^{\bullet}r, \quad r^{\bullet} = Hr, \quad H = \frac{1}{\mathcal{K}}\mathcal{K}^{\bullet}, \tag{6.2}$$

with H as Hubble parameter. With (4.5) again results (6.1). The horizon of the model is therefore at $r = \mathcal{R}_0$, ie at the equator of the spherical space which is assigned to any observer at $r = 0$.

II. Similarly we proceed with the *Lanczos model*. One has $\mathcal{K} = ch\psi'$, $r = \mathcal{K}r'$ and the Hubble parameter is

$$H = \frac{1}{\mathcal{K}}\mathcal{K}^{\bullet} = \frac{1}{\mathcal{R}_0}th\psi', \quad r = \mathcal{R}_0\sin\eta.$$

Thus, with (4.12) one has for the *coordinate velocity*

$$r^{\bullet} = Hr = \frac{r}{\mathcal{R}_0} \operatorname{th}\psi' = \sin\eta \, \operatorname{th}\psi' = \sin\eta' \operatorname{sh}\psi' . \tag{6.3}$$

The *relative velocity* in the expanding universe of a point calculated from the view of a non-comoving observer is

$$v = \frac{dx^1}{dT}, \quad dx^1 = \overset{1}{e}_1 dr = \frac{1}{\cos\eta} dr, \quad dT = \alpha dT' = \frac{\cos\eta'}{\cos\eta} dt' .$$

Thus, is

$$v = \frac{1}{\alpha\cos\eta} r^{\bullet} = \frac{1}{\cos\eta'} r^{\bullet} , \qquad ,$$

$$v = \tan\eta' \operatorname{sh}\psi' = \sin\eta \, \operatorname{th}\psi , \tag{6.4}$$

an expression that we also can gather from the transformation (5.5). Since the relative velocity reaches the velocity of light as its highest value, (5.5) is a Lorentz transformation.

III. In the *Lanczos-like cosmos* is $\mathcal{K} = \operatorname{sh}\psi'$, $r = \mathcal{K} r'$ and the Hubble parameter is

$$H = \frac{1}{\mathcal{K}} \mathcal{K}^{\bullet} = \frac{1}{\mathcal{R}_0} \operatorname{ch}\psi', \quad r = \mathcal{R}_0 \operatorname{sh}\eta .$$

The *coordinate* velocity

$$r^{\bullet} = Hr = \frac{r}{\mathcal{R}_0} \operatorname{ch}\psi' = \operatorname{sh}\eta \, \operatorname{cth}\psi' = \operatorname{sh}\eta' \operatorname{ch}\psi' \tag{6.5}$$

can be infinitely high. The relative velocity is calculated with

$$v = \frac{dx^1}{dT}, \quad dx^1 = \overset{1}{e}_1 dr = \frac{1}{\sqrt{1-\operatorname{sh}^2\eta}} dr, \quad dT = \alpha dT' = \frac{\operatorname{ch}\eta'}{\sqrt{1-\operatorname{sh}^2\eta}} dt'$$

$$v = \operatorname{th}\eta' \operatorname{ch}\psi' = \operatorname{sh}\eta \, \operatorname{cth}\psi \tag{6.6}$$

in accordance with (5.8). Both the coordinate velocity and the relative velocity can be infinitely high. Thus, the pseudo-rotation (5.8) does not correspond to the conditions required for a Lorentz transformation.

IV. For the *anti-de Sitter universe* counts

$$\mathcal{K} = \sin\psi', \quad r = \mathcal{K} r', \quad r = \mathcal{R}_0 \operatorname{sh}\eta, \quad H = \frac{1}{\mathcal{R}_0} \cot\psi'$$

and therefore for the *coordinate* velocity

$$r^{\bullet} = Hr = \operatorname{sh}\eta \cot\psi' = \operatorname{sh}\eta' \cos\psi' . \tag{6.7}$$

It is infinitely high at infinity of the open universe. With

$$\overset{1}{e}_1 = \frac{1}{\mathrm{ch}\eta}, \quad \alpha = \frac{\mathrm{ch}\eta'}{\mathrm{ch}\eta}, \quad v = \frac{1}{\mathrm{ch}\eta} r^{\bullet}$$

the *relative velocity* is

$$v = \mathrm{th}\eta' \cos\psi' = \mathrm{sh}\eta \cot\psi \tag{6.8}$$

in accordance with (5.12). Its highest value lies at infinity and is infinitely high.

We clearly present the results in a table.

I	$0 \le r \le \mathcal{R}_0$	$r = \mathcal{R}_0 \sin\eta$	$\mathcal{K} = e^{\psi'}$	$v = \frac{r}{\mathcal{R}_0}$	$v_{max} = c$	Lorentz transformation
II	$0 \le r \le \mathcal{R}_0$	$r = \mathcal{R}_0 \sin\eta$	$\mathcal{K} = \mathrm{ch}\psi'$	$v = \frac{r}{\mathcal{R}_0} \mathrm{th}\psi$	$v_{max} = c$	Lorentz transformation
III	$0 \le r \le \infty$	$r = \mathcal{R}_0 \mathrm{sh}\eta$	$\mathcal{K} = \mathrm{sh}\psi'$	$v = \frac{r}{\mathcal{R}_0} \mathrm{cth}\psi$	$v_{max} = \infty$	pseudo-rotation
IV	$0 \le r \le \infty$	$r = \mathcal{R}_0 \mathrm{sh}\eta$	$\mathcal{K} = \sin\psi'$	$v = \frac{r}{\mathcal{R}_0} \cot\psi$	$v_{max} = \infty$	pseudo-rotation

It is evident that in the de Sitter cosmos and in the Lanczos cosmos the relative velocity can accept the velocity of light as its highest value, while the relative velocity for the other two models is unlimited. With these models the pseudo-rotation does not satisfy the conditions of a Lorentz transformation. The universes I and II are closed and are in their structures a little close to our universe. The two open-infinite universes III and IV are not suitable for trial description of our universe.

We note further that the definition of the relative velocity differs from the definition of the recession velocity, as is customary in FRW models. In FRW models the distance between two points in space is often defined by

$$l = \mathcal{R}\eta'.$$

This results in the Hubble law

$$l^{\bullet} = Hl, \quad H = \frac{\mathcal{R}^{\bullet}}{\mathcal{R}} = \frac{\mathcal{K}^{\bullet}}{\mathcal{K}}, \tag{6.9}$$

which allows superluminal velocity for the recession velocity. Both the two definitions $r^{\bullet} = Hr$ and $l^{\bullet} = Hl$ describe the Hubble law, but have different implications. However, the two definitions refer to coordinate speeds which cannot reflect the motion correctly.

It should be noted that the relative velocity, which results from the Lorentz transformation between the two systems is in accordance with the definitions of other gravity models. Thus, Lemaitre has established a coordinate transformation for the Schwarzschild theory, which is related to the free fall. This coordinate transformation can be assigned to a Lorentz transformation [8], from which can be taken the fall velocity. Thereby, the geometrical distance between two points on Flamm's paraboloid ($l = ar + M \ln\frac{1+a}{1-a}$, $a = \sqrt{1-2M/r}$) does not matter. All the calculations are handled with the coordinate r in the flat embedding space. The definition (6.9) common in the Friedman model is clearly isolated.

7. CONCLUSIONS

For the four cosmological models of the de Sitter family have been assigned pseudo-rotations to coordinate transformations between observers who comove with the expansion and observers who do not comove. Relative velocities have been read from these pseudo-rotations. In the de Sitter model and the Lanczos model values obtained for the velocities are in the relativistic range. Infinitely high relative velocities are possible for the Lanczos-like and anti-de Sitter universe. In this case the pseudo-rotations cannot be interpreted as Lorentz transformations. Therefore, pseudo-rotations classify the models as physically suitable or unsuitable. The closed models do not admit superluminal velocities for the recession velocities of galaxies. The open, infinite models allow arbitrarily high velocities. However, with open models the transformations between co-moving and non-comoving observers violate the fundamental laws of special relativity.

We will calculate the field quantities and field equations of all these models in a subsequent paper.

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