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FREE FALL AND TIME FUNCTION

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Contents

Abstract: We investigate the free fall in the Schwarzschild field and we calculate the time function for observers falling in from infinity or from an arbitrary finite position. We show that any observer can only reach the event horizon in infinite proper time.

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1. INTRODUCTION

In recent decades, the problem of free fall has repeatedly been treated in the literature. While general agreement exists that an observer who comes from infinity would reach the speed of light at the event horizon, there are controversial views regarding the speed of an infalling observer who falls from a finite position towards the center of gravitation. We attack the problem and we will show that an observer coming from infinity or from any other position would attain the speed of light at the event horizon and therefore will take an infinitely long proper time. To manage this problem one has to apply Einstein’s addition law of velocities and the velocity formula for the free fall in the Schwarzschild field. We [1,2,3] have investigated this problem in some previous papers.

2. THE FALL FROM THE INFINITE

We are confronted with the actual problem that initially only the velocity of an observer B'' who comes from infinity is known. It is determined by the Schwarzschild geometry by

$$v = v(r) = -\sqrt{\frac{2M}{r}} \qquad (2.1)$$

and therefore is a geometric quantity. To simplify the following consideration, we reduce the problem to a 2-dimensional one. We suppress the ϑ- and φ-dimensions and we denote the radial co-ordinate with x.

The observer B'' coming from the infinite does not change his position in the comoving system. Therefore, one has

$$\frac{dx''}{dT''} = 0, \qquad x'' = \text{const.} \;, \qquad (2.2)$$

where T'' is the proper time of B''. In view of the system B, which is in rest, his velocity is

$$\frac{dx}{dT} = v \;, \qquad (2.3)$$

where we now have used the proper time T of the static system.

If we include the well-known relation

$$\frac{dT}{dT''} = \alpha \qquad (2.4)$$

with α as the Lorentz factor of the transformation $x \leftrightarrow x''$ and if we take into account the relation $dx = \alpha dr$, we have

$$v = \frac{dr}{dT''}, \quad dT'' = \frac{1}{v} dr \;. \qquad (2.5)$$

The integral of dT''

$$T''(r) = -\int \sqrt{\frac{r}{2M}}dr = -\frac{1}{3}\sqrt{\frac{2r^3}{M}} \tag{2.6}$$

is a well-known expression in the literature which graphically shows a curve being zero at $r = 0$ and increasing to infinity for $r \to \infty$. It determines the time the observer has to take in order to reach a point r, starting at $r = 0$. Since there is invariance under time reversal, one obtains for the fall time a function that is infinitely high at $r = 0$.

It is noticeable that the curve $T''(r)$ crosses the event horizon, although any incoming object would reach the speed of light at this location. The fall velocity $v(r) = -\sqrt{2M/r}$ is mathematically continued into the inner region $0 < r < 2M$ of the Schwarzschild solution. Thus, it is mathematically quite correct that the integral (2.6) also covers the inner region. Due to our geometrical interpretation of the gravitation theory a penetration of the event horizon is not possible. The circle at the throat of Flamm's paraboloid is the boundary of the geometry and beneath it no statements can be made.

This raises the question whether the integral (2.6) can be corrected in such a way that an observer incoming from infinity requires an infinitely long proper time to reach the event horizon. In fact, the problem can be solved easily, if one calculates the integral (2.6) within limits. From[1]

$$T''(r) = \int_{2M}^{r} \sqrt{\frac{r}{2M}}dr = \frac{1}{3}\sqrt{\frac{2r^3}{M}} - \frac{1}{3}4M \tag{2.7}$$

one obtains the rise time for an observer from 2M to infinity, whereas

$$T''(2M) = 0, \quad \lim_{r\to\infty} T''(r) = \infty \tag{2.8}$$

is valid. Due to the invariance under time reversal, an observer who comes from the infinite reaches the event horizon only after infinite proper time.

Since the event horizon is considered to be unreachable, neither for a falling observer nor for the surface of a collapsing stellar object, the existence of black holes is questioned. One might argue that the restriction of the integral (2.7) to the range $[2M \le r \le \infty]$ is arbitrary. To justify this practice, we calculate the problem once again with other variables viz with those variables that can easily be taken from the Schwarzschild geometry, but which are defined from the beginning above the event horizon. Furthermore, we extend the investigation to the case where the incoming object does not come from infinity, but from any finite position.

The fall velocity of an object coming from infinity is a quantity closely associated with the Schwarzschild geometry. It is related to the angle of ascent of the Schwarzschild parabola by

$$v = \sin\varepsilon = -\sqrt{2M/r} \ . \tag{2.9}$$

By rearranging and differentiating this relation we obtain

$$r = \frac{2M}{\sin^2\varepsilon}, \quad dr = -\frac{4M}{\sin^3\varepsilon}\cos\varepsilon \, d\varepsilon \ . \tag{2.10}$$

[1] For reasons of clarity we use here the value of the fall velocity.

The geometry at infinity is flat, because of $\varepsilon = 0$ at this position. The tangent at the vertex of the Schwarzschild parabola is normal to r-axis, i.e. $\varepsilon = -\pi/2$. From

$$dT'' = \frac{1}{v}dr = -\frac{4M}{\sin^4\varepsilon}\cos\varepsilon\, d\varepsilon \tag{2.11}$$

one obtains with the lower limit $-\pi/2$ the function[2]

$$T''(\varepsilon) = \frac{4M}{3\sin^3\varepsilon} + \frac{4M}{3} \ , \tag{2.12}$$

which describes the rise time. Due to the invariance under time reversal, one obtains the curve for the fall time. If the fall velocity is parameterized with the angle of ascent of the Schwarzschild parabola one obtains for the time function a curve starting with zero value at $\varepsilon = 0$ and increasing to the infinite at the event horizon. The choice of this variable has the advantage that the spatial infinity can be represented graphically. The same applies to isotropic co-ordinates, which have been treated by us in a former paper [4] in more detail.

From the nonlinear transformation

$$r = \left(1 + \frac{M}{2\bar{r}}\right)^2 \bar{r} \tag{2.13}$$

one obtains

$$dr = \left(1 - \frac{M^2}{4\bar{r}^2}\right) d\bar{r} \ . \tag{2.14}$$

The new isotropic co-ordinate $\bar{r}$ describes both branches of the Schwarzschild parabola, by running through them. It starts at $\bar{r} = 0$ at infinity on the lower branch of the parabola and runs into the infinite of the upper branch ($\bar{r} = \infty$). The co-ordinate has a minimum at $\bar{r} = M/2$ which corresponds to $r = 2M$, the event horizon. Therefore isotropic co-ordinates describe only the outer region $r \geq 2M$ of the Schwarzschild geometry, but in a twofold manner. The fall velocity is

$$v(\bar{r}) = -\frac{1}{1 + \frac{M}{2\bar{r}}}\sqrt{\frac{2M}{\bar{r}}} \ . \tag{2.15}$$

Taking into account (2.5) together with (2.14) and (2.15) one arrives at

$$dT'' = -\frac{8\bar{r}^3 + 4M\bar{r}^2 - 2M^2\bar{r} - M^3}{8\bar{r}^3}\sqrt{\frac{\bar{r}}{2M}}d\bar{r} \ . \tag{2.16}$$

If we take the positive value of v for the sake of simplicity, the integral of (2.16) provides the function

$$f(r) = \frac{1}{4\sqrt{2M}}\left[\frac{8}{3}\sqrt{\bar{r}^3} + 4M\sqrt{\bar{r}} + \frac{2M^2}{\sqrt{\bar{r}}} + \frac{M^3}{3\sqrt{\bar{r}^3}}\right] . \tag{2.17}$$

Within the limits of the lower branch one has

$$T''(\bar{r}) = f(\bar{r}) - f\left(\frac{M}{2}\right) \tag{2.18}$$

[2] The first term is negative because ε is negative.

with $\lim_{\bar{r}\to 0} T''(\bar{r}) = \infty$ and $T''\left(\frac{M}{2}\right) = 0$. Time symmetry attributes to the function for the fall time, which can be seen from Fig. 2.1 ($M = 2$).

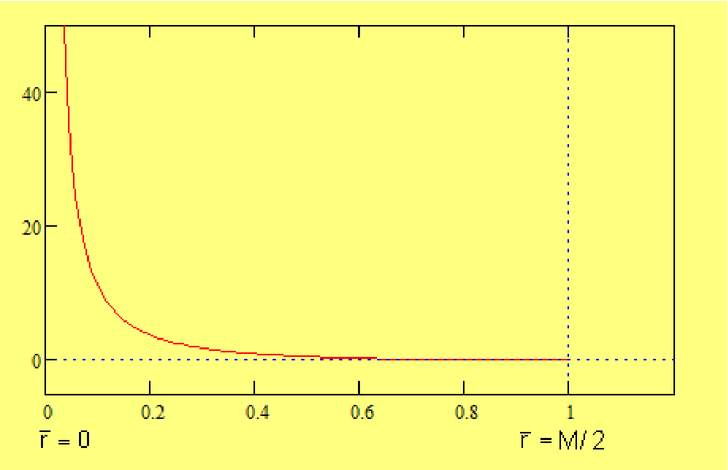

Fig. 2.1

One obtains the same results with Einstein-Rosen co-ordinates which we have treated in [2] and also with the Lorentz angle as parameter. The latter problem will be executed in Section 6. By using different variables, we have shown in this Section that an observer infalling from the infinite can reach the event horizon only after infinite proper time. Whether this also applies to objects coming from any finite position, will be examined in the following.

3. FREE FALL FROM AN ARBITRARY POSITION

The velocity of an observer who is infalling from an arbitrary position can be determined only circuitously. For this purpose, we perform the following considerations:

An object coming from infinity has at an arbitrary position r_0 the velocity $v_0 = -\sqrt{2M/r_0}$. Another object is released from r_0 at the very moment the first object is passing the point r_0. In this moment the difference of their fall velocity is simply v_0. However, the difference decreases according to Einstein's composition law of velocities. With regard to the static Schwarzschild system the speed of the second object is calculated with respect to the relative velocity of the first object

$$v' = v(r, r_0) = \frac{-\sqrt{\frac{2M}{r}} - \left(-\sqrt{\frac{2M}{r_0}}\right)}{1 - \sqrt{\frac{2M}{r}}\sqrt{\frac{2M}{r_0}}} \; . \tag{3.1}$$

The latter is at the starting position $v(r_0, r_0) = 0$, at the event horizon $v(2M, r_0) = -1$. Fig. 3.1 shows some examples. The top curve corresponds to the observer who comes from infinity.

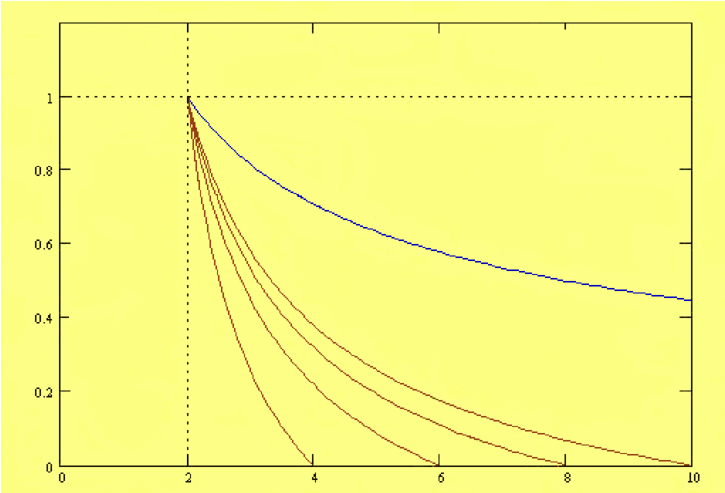

Fig. 3.1. Velocities

With these considerations, we have left behind the possibility that incoming objects can cross the Schwarzschild radius and the hypothesis of black holes.

To handle the velocity relations we introduce three reference frames. The first is called B being in rest in the Schwarzschild field, the second is the reference system B' which accompanies an observer with the velocity v', and the third B" is coming in free fall from infinity with the velocity v. The systems are connected by the Lorentz relations

$$v' = \frac{v - v_0}{1 - v v_0}, \qquad v = \frac{v' + v_0}{1 + v' v_0}, \qquad v_0 = \frac{v - v'}{1 - v v'} , \tag{3.2}$$

$$\alpha' = \alpha\alpha_0 \left(1 - v v_0\right), \quad \alpha = \alpha' \alpha_0 \left(1 + v' v_0\right), \quad \alpha_0 = \alpha' \alpha \left(1 - v' v\right) , \tag{3.3}$$

$$\alpha' v' = \alpha\alpha_0 \left(v - v_0\right), \quad \alpha v = \alpha' \alpha_0 \left(v' + v_0\right), \quad \alpha_0 v_0 = \alpha\alpha' \left(v - v'\right) . \tag{3.4}$$

The quantities α are the Lorentz factors associated with the relative velocities. With the help of a table we provide other useful relations.

The Lorentz transformations have the form

$$\begin{aligned} &L_1^{1'} = \alpha', \quad L_1^{4'} = -i\alpha' v', \quad L_4^{1'} = i\alpha' v', \quad L_4^{4'} = \alpha' \\ &L_{1'}^{1''} = \alpha_0, \quad L_{1'}^{4''} = -i\alpha_0 v_0, \quad L_{4'}^{1''} = i\alpha_0 v_0, \quad L_{4'}^{4''} = \alpha_0 \quad . \\ &L_1^{1''} = \alpha, \quad L_1^{4''} = -i\alpha v, \quad L_4^{1''} = i\alpha v, \quad L_4^{4''} = \alpha \end{aligned} \tag{3.5}$$

For the table, we calculate the relative velocities and proper times of the above-mentioned three observers. The position of an observer coming from infinity does not change with respect to the comoving observer system. For B" one has $x'' = \text{const.}$, $dx'' = 0$.

Next, we write $dx^{4''} = i dT''$, $dx^{4'} = i dT'$, where dT'' and dT' are the proper times of the observers with respect to B" and B'. From the Lorentz transformation

$$dx^{1''} = L_{1'}^{1''} dx^{1'} + L_{4'}^{1''} dx^{4'}, \quad dx^{4''} = L_{1'}^{4''} dx^{1'} + L_{4'}^{4''} dx^{4'}$$

we infer the relation $0 = \alpha_0 dx' + i\alpha_0 v_0 dT'$ which leads to

$$\frac{dx'}{dT'} = v_0 .$$

I. $x''=\text{const.}$						
systems	L	transformations	rel. velocities	phys. time	rel. vel. of	meas. in
1. B'' || B'	L(v_0)	$dx^{m''}=L^{m''}_{m'}dx^{m'}$	$\frac{dx'}{dT'}=v_0$	$\frac{dT'}{dT''}=\alpha_0$	B'' a .B'	B'
2. B'' || B	L(v)	$dx^{m''}=L^{m''}_{m}dx^{m}$	$\frac{dx}{dT}=v$	$\frac{dT}{dT''}=\alpha$	B'' a. B	B
3. B || B'	L(v')	$dx^{m}=L^{m}_{m'}dx^{m'}$	$\frac{dx''}{dT''}=0$	$\frac{dT}{dT'}=\frac{\alpha}{\alpha_0}$		
II. $x'=\text{const.}$						
systems	L	transformations	rel. velocities	phys. time	rel. vel. of	meas. in
1. B' || B	L(v')	$dx^{m'}=L^{m'}_{m}dx^{m}$	$\frac{dx}{dT}=v'$	$\frac{dT}{dT'}=\alpha'$	B' a. B	B
2. B' || B''	L(v_0)	$dx^{m'}=L^{m'}_{m''}dx^{m''}$	$\frac{dx''}{dT''}=-v_0$	$\frac{dT''}{dT'}=\alpha_0$	B' a. B''	B''
3. B'' || B	L(v)	$dx^{m''}=L^{m''}_{m}dx^{m}$	$\frac{dx'}{dT'}=0$	$\frac{dT''}{dT}=\frac{\alpha_0}{\alpha'}$		
III. $x=\text{const.}$						
systems	L	transformations	rel. velocities	phys. time	rel. vel. of	meas. in
1. B || B'	L(v')	$dx^{m}=L^{m}_{m'}dx^{m'}$	$\frac{dx'}{dT'}=-v'$	$\frac{dT'}{dT}=\alpha'$	B a. B'	B'
2. B || B''	L(v)	$dx^{m}=L^{m}_{m''}dx^{m''}$	$\frac{dx''}{dT''}=-v$	$\frac{dT''}{dT}=\alpha$	B a. B''	B''
3. B' || B''	L(v_0)	$dx^{m'}=L^{m'}_{m''}dx^{m''}$	$\frac{dx}{dT}=0$	$\frac{dT'}{dT''}=\frac{\alpha'}{\alpha}$		

In addition, one has with $idT''=-i\alpha_0 v_0 dx'+\alpha_0 idT'$ and with the result obtained above

$$dT''=\alpha_0\left(-v_0^2+1\right)dT'$$

and finally

$$\frac{dT'}{dT''}=\alpha_0 .$$

Similar considerations can be made for all rows of the table and we obtain the elementary relations listed in it of which we frequently make use. With these relations and the velocity definition (3.1) we are prepared to study the free fall of objects falling from a finite position.

4. SCHWARZSCHILD STANDARD CO-ORDINATES

In the preceding Sections we have examined the free fall from infinity in several respects. Let us now consider the more complicated case in which an observer can fall away from any position. In the last Section we have determined the required velocity definitions and we have prepared the essential mathematics.

With the help of the table of Section 3, we refer to an observer B' who falls in from an arbitrary position r_0, and we gather the relations

$$\frac{dx}{dT} = v', \quad \frac{dT}{dT'} = \alpha', \quad x' = \text{const.} \ . \tag{4.1}$$

With $dx = \alpha dr$ we write

$$\frac{\alpha dr}{dT'} = \alpha' v' \ ,$$

and we bear in mind that the metric coefficient α is identical with the Lorentz factor of an observer incoming from the infinite. With (3.4) one has

$$dT' = \frac{\alpha}{\alpha' v'} dr = \frac{1}{\alpha_0 \left(v - v_0\right)} dr \ . \tag{4.2}$$

The integration of this expression leads to an integral of the type

$$\int \frac{\sqrt{x}}{\sqrt{x} - 1} dx = x + 2\sqrt{x} + 2\ln(1 - \sqrt{x}), \quad x < 1, \quad \lim_{x \to 1} \int \frac{\sqrt{x}}{\sqrt{x} - 1} dx = \infty \tag{4.3}$$

and with $x = \dfrac{r}{r_0}$ to the function

$$f\left(r, r_0\right) = -\sqrt{\frac{r_0}{2M} - 1} \left[r - 2\sqrt{r_0 r} + 2 r_0 \ln\left(1 - \sqrt{\frac{r}{r_0}}\right)\right] \tag{4.4}$$

which describes the rise time, and which is infinite at r_0. As an observer ascending from the event horizon cannot be realized physically, we will mirror the function. Substituting $r \to r_0 - r$ into the function (4.4), we measure the instantaneous fall distance beginning at r_0. With $r \to r - 2M$, $\bar{r}_0 \to r_0 - 2M$ we restrict the integration to the range $\left[2M, r_0\right]$. Thus, we get from (4.4) a function which is plotted in Fig. 4.1. One can see that for the starting point and end point of the downward motion the relations

$$T'\left(r_0 - r, r_0 - 2M\right)_{|r = r_0} = 0, \quad \lim_{r \to 2M} T'\left(r_0 - r, r_0 - 2M\right) = \infty \tag{4.5}$$

are valid. A freely falling observer can reach the event horizon only after an infinite proper time.

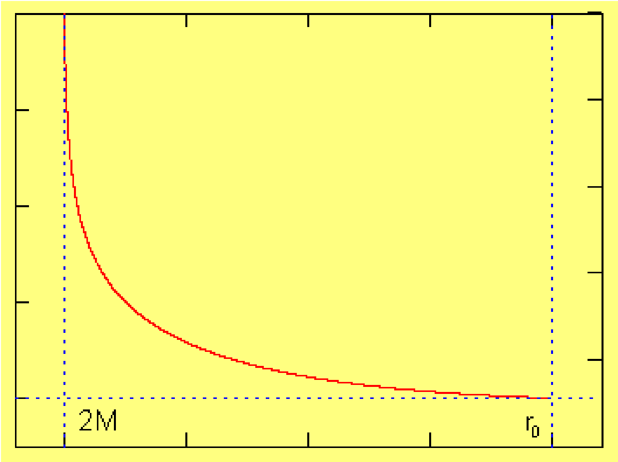

Fig. 4.1. Fall time

In order to free ourselves from the suspicion that we have restricted the variable r arbitrarily in order to exclude the region $r < 2M$, we carry out the same calculations again with other variables. We use variables which in principle cannot describe the inner region of the Schwarzschild geometry.

5. CALCULATION WITH THE ANGLE OF ASCENT

We extend the method discussed in Section 2, where we have used the angle of ascent of the Schwarzschild parabola as parameter. For an observer coming from the infinite, we use for the fall velocity the quantity (3.1). For the velocity of an observer who comes from infinity and has reached the position r_0 we find the expression

$$v_0 = \sin\varepsilon_0 = -\sqrt{2M/r_0} \quad , \tag{5.1}$$

where ε_0 is a negative angle.

If we insert this into (3.1) we have

$$v' = v(\varepsilon, \varepsilon_0) = \frac{\sin\varepsilon - \sin\varepsilon_0}{1 - \sin\varepsilon \sin\varepsilon_0} \quad . \tag{5.2}$$

T' is the proper time of a freely falling observer who comes from a finite position. The Lorentz factor for the constant velocity v_0 is

$$\alpha_0 = \frac{1}{\sqrt{1 - \sin^2\varepsilon_0}} = \frac{1}{\cos\varepsilon_0} \quad . \tag{5.3}$$

Then (4.2) becomes with (2.10)

$$dT' = \frac{\cos\varepsilon_0}{\sin\varepsilon - \sin\varepsilon_0}\left(-\frac{4M}{\sin^3\varepsilon}\cos\varepsilon\, d\varepsilon\right) \quad . \tag{5.4}$$

To make the calculations clearer, we take the angle to be positive. In the region $\varepsilon=\left[\frac{\pi}{2},\varepsilon_0\right]$ we have $\varepsilon>\varepsilon_0$. Integration leads to

$$f(\varepsilon)=-4M\cos\varepsilon_0\left[\frac{1}{2\sin^2\varepsilon\sin\varepsilon_0}+\frac{1}{\sin\varepsilon\sin^2\varepsilon_0}-\frac{1}{\sin^3\varepsilon_0}\ln\frac{\sin\varepsilon}{\sin\varepsilon-\sin\varepsilon_0}\right]. \tag{5.5}$$

Within the limits one has

$$T'(\varepsilon)=f(\varepsilon)-f\left(\frac{\pi}{2}\right),\quad T'\left(\frac{\pi}{2}\right)=0,\quad \lim_{\varepsilon\to\varepsilon_0}T'(\varepsilon)=\infty \tag{5.6}$$

Time symmetry leads back to the previously developed results as shown in Fig. 5.1.

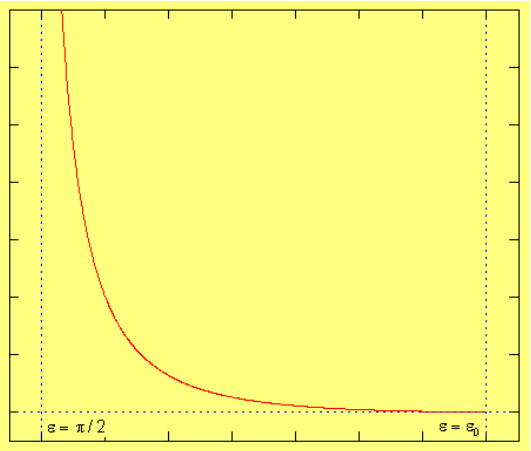

Fig. 5.1. Fall time

6. CALCULATION WITH THE LORENTZ ANGLE

The velocity of an observer falling freely from infinity can be written as

$$v=-\text{th}\chi \tag{6.1}$$

as well. The Lorentz angle χ is also called rapidity. For $\chi\to\infty$ one gets $v=-1$. For the Lorentz factor one has $\alpha=\text{ch}\chi$. With

$$i\alpha v=-i\,\text{th}\chi\,\text{ch}\chi=-i\,\text{sh}\chi=-\sin i\chi,\quad \alpha=\cos i\chi$$

we notice that with (6.1) a rotation through the angle $i\chi$ is finally described, i.e. a pseudo-rotation is applied. With

$$v'=\frac{v-v_0}{1-vv_0}=\frac{\text{th}\chi-\text{th}\chi_0}{1-\text{th}\chi\,\text{th}\chi_0}=\text{th}(\chi-\chi_0) \tag{6.2}$$

one obtains the simple relation

$$\text{th}\chi'=\text{th}(\chi-\chi_0),\quad \chi>\chi_0. \tag{6.3}$$

In addition, one has from (2.1)

$$dr = -\frac{4M}{v^3} dv$$

and with (6.1)

$$dv = -\frac{1}{ch^2\chi} d\chi \ .$$

One finally gets

$$dr = -\frac{4M}{sh^2\chi \, th\chi} d\chi \ .$$

With

$$dT' = \frac{\alpha}{\alpha' v'} dr$$

one obtains

$$dT' = \frac{4M}{sh(\chi - \chi_0)} \frac{1}{sh\chi \, th^2\chi} d\chi \ . \qquad (6.4)$$

The integral yields

$$f(\chi) = \frac{2M}{sh\chi_0}\left[2 cth\chi \, cth\chi_0 + \frac{1}{sh^2\chi} + 2 cth^2\chi_0 \ln\frac{sh(\chi - \chi_0)}{sh\chi}\right] \ . \qquad (6.5)$$

For $\chi \to \infty$, i. e., at the location $r = 2M$, we obtain

$$g = 4M \frac{cth\chi_0}{sh\chi_0}(1 - \chi_0 cth\chi_0) \ ,$$

so that we finally gain the time function

$$T'(\chi) = f(\chi) - g, \quad \lim_{\chi\to\infty} T'(\chi) = 0, \quad \lim_{\chi\to\chi_0} T'(\chi) = \infty \ . \qquad (6.6)$$

Due to time reversal we obtain the fall time again.

7. CONCLUSIONS

From the previous considerations conclusions can be drawn for the collapse of a star. If the surface of a stellar object is located at the position r_0 , and even if one assumes the highest possible speed of contraction, namely the free fall, the object can never contract to the event horizon, or even exceed it. This has far-reaching consequences for the theory of stellar collapse. Models that satisfy this condition have been proposed by Mitra [5].

We have shown that an observer in the Schwarzschild field infalling from infinity or from any other position can reach the event horizon only after infinite proper time. We have taken up a position in contrast to the current literature and we will show elsewhere how it can lead to different interpretations concerning the velocities and fall times of free fall.

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