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EINSTEIN'S ELEVATOR

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Contents

Abstract: We calculate the field strengths for an observer falling in the Schwarzschild field from an arbitrary position towards the gravitation center. We show that two forces emerge nullifying the counterforce due to the acceleration of the observer. One force is related to the gravitational force acting on an observer in rest with respect to the gravitation center, the other acting on an observer coming from the infinite.

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1. INTRODUCTION

In previous papers [1,2] we have demonstrated that all observers who fall from an arbitrary position in the Schwarzschild field to the center of gravity reach the speed of light at the event horizon. Now we will examine the acceleration and the forces acting on an observer who starts at a position $r_0 > 2M$. In our previous papers we already have noted that an observer who comes from the infinite experiences no gravity effect. Here we will work out that this is the case for observers incoming from an arbitrary position as well.

2. FORCES AND LORENTZ TRANSFORMATIONS

It was Einstein's idea that in a curved space a co-ordinate system can be chosen in such a way that the space appears flat at any point or along a curve, i.e. that an observer can experience no forces in this system. In modern language this means that a family of local reference systems can be chosen which accompany such an observer and that the observer does not experience any forces with respect to these systems. In the Schwarzschild case, this means that the observer is moving with a velocity $v = v(r)$, where v is the velocity of a freely falling observer who comes from the infinite. This problem can be treated quite simply, because the speed v can be derived from the Schwarzschild metric and the acceleration associated with the speed is easy to calculate. The situation is different if we allow an object to start from a position r_0. The associated velocity v' can only intricately be derived from the (relativistic) difference of two velocities, and so it will be with the forces.

To get the problem in hand we must refer to an observer who comes from infinity and has the speed $v_0 = -\sqrt{2M/r_0}$ in passing the position r_0. At the very moment at which the observer reaches r_0, a second observer is released from there and also moves in free fall to the center of gravity. We have discussed his velocity in our paper [3] by using Einstein's addition theorem of velocities and the velocity $v_0 = \text{const.}$ for reference. This will now be discussed in greater detail by using Lorentz transformations. This enables us to derive the forces that act on these falling observers.

We use the following basic idea: physical parameters in two local reference systems that relate to the same event are connected by a Lorentz transformation, even if one or both of these reference systems accelerate. The reason is that the length scales and time intervals do not depend on the acceleration but only on the speed. Applying to our problem, this means that we face three reference systems. The static system B which is in rest relative to the center of gravity, the system B' which falls in from r_0 with increasing velocity v', and finally the system B'' which is coming from infinity in free fall and moves towards the center of gravity with the velocity v. Thus, the velocities and the associated Lorentz factors are

$$v' = \frac{v - v_0}{1 - v v_0}, \qquad v = \frac{v' + v_0}{1 + v' v_0}, \qquad v_0 = \frac{v - v'}{1 - v v'}, \qquad (2.1)$$

$$\alpha' = \alpha\alpha_0\left(1 - vv_0\right), \quad \alpha = \alpha'\alpha_0\left(1 + v'v_0\right), \quad \alpha_0 = \alpha'\alpha\left(1 - v'v\right), \tag{2.2}$$

$$\alpha'v' = \alpha\alpha_0\left(v - v_0\right), \quad \alpha v = \alpha'\alpha_0\left(v' + v_0\right), \quad \alpha_0 v_0 = \alpha\alpha'\left(v - v'\right), \tag{2.3}$$

which one can conclude from the Lorentz transformations

$$\begin{array}{llll} L_1^{1'} = \alpha', & L_1^{4'} = -i\alpha'v', & L_4^{1'} = i\alpha'v', & L_4^{4'} = \alpha' \\ L_{1'}^{1"} = \alpha_0, & L_{1'}^{4"} = -i\alpha_0 v_0, & L_{4'}^{1"} = i\alpha_0 v_0, & L_{4'}^{4"} = \alpha_0 \\ L_1^{1"} = \alpha, & L_1^{4"} = -i\alpha v, & L_4^{1"} = i\alpha v, & L_4^{4"} = \alpha \end{array} \tag{2.4}$$

which connect these three systems.

Both v and v' take the values -1 at the event horizon, i.e. the speed of light. The associated Lorentz factors are infinitely high at this position. α takes the value 1 at infinity, but α' at the position r_0. The velocity $v = -\sqrt{2M/r}$ of a freely falling observer is given by the Schwarzschild geometry. It corresponds to $\sin\varepsilon$, ε being the angle of ascent of the Schwarzschild parabola. But v' can only be accessed by the first formula (2.1) in using the reference speed v_0.

First, we derive some useful relations. Differentiating the second equation (2.1) and (2.2) gives

$$\alpha^2 dv = \alpha'^2 dv' + \alpha_0^2 dv_0 , \tag{2.5}$$

$$\frac{1}{\alpha} d\alpha v = \frac{1}{\alpha'} d\alpha'v' + \frac{1}{\alpha_0} d\alpha_0 v_0 , \tag{2.6}$$

wherein each of the last two terms vanishes if one keeps v_0 constant. If one denotes the proper times of the three systems with T, T ', T", one gets from (2.6) the accelerations of the observers

$$\frac{1}{\alpha}\frac{d\alpha v}{dT"} = G_{1"}, \quad \frac{1}{\alpha'}\frac{d\alpha'v'}{dT'} = G_{1'} . \tag{2.7}$$

The indices of the quantities G denote the radial components of the quantities in the corresponding reference systems. These are the systems which are comoving with the observers B" and B '.

In addition, multiplying with the rest mass m_0 we obtain the Lorentz forces

$$\frac{1}{\alpha}\frac{dmv}{dT"} = K_{1"}, \quad \frac{1}{\alpha'}\frac{dm'v'}{dT'} = K_{1'}, \quad m = m_0\alpha, \quad m' = m_0\alpha' \tag{2.8}$$

in accordance with the definitions of the electrodynamics of moving media. In these relations the proper time of the static system can also be used. From

$$\begin{aligned} dx^{1"} &= L_m^{1"} dx^m = \alpha dx^1 + i\alpha v\, dx^4 \\ dx^{1'} &= L_m^{1'} dx^m = \alpha' dx^1 + i\alpha' v' dx^4 \end{aligned} \tag{2.9}$$

one obtains with $dx^{4"} = idT"$, $dx^{4'} = idT'$, $dx^4 = idT$ the relative velocities with respect to the system in rest

$$\frac{dx^1}{dT} = v,\ x^{1''} = \text{const.},\quad \frac{dx^1}{dT} = v',\ x^{1'} = \text{const.}\ . \tag{2.10}$$

Further, from

$$\begin{aligned} dx^{4''} &= L^{4''}_{m} dx^m = -i\alpha v dx^1 + \alpha\, dx^4, \quad dT'' = \alpha\left(dT - v\, dx^1\right) = \frac{1}{\alpha} dT \\ dx^{4'} &= L^{4'}_{m} dx^m = -i\alpha' v' dx^1 + \alpha' dx^4, \quad dT' = \alpha'\left(dT - v' dx^1\right) = \frac{1}{\alpha'} dT \end{aligned} \tag{2.11}$$

the relations

$$\frac{dT}{dT''} = \alpha,\ x^{1''} = \text{const.},\quad \frac{dT}{dT'} = \alpha',\ x^{1'} = \text{const.}\ . \tag{2.12}$$

Inserting into (2.8) one has

$$\frac{dmv}{dT} = K_{1''}\quad \frac{dm'v'}{dT} = K_{1'}\ . \tag{2.13}$$

These quantities can be calculated, because one can deduce the velocities and Lorentz factors from the Schwarzschild geometry. From

$$\Phi_{|4''} = L^{m}_{4''}\Phi_{|m},\quad \Phi_{|4'} = L^{m}_{4'}\Phi_{|m}$$

one reads

$$\frac{d\Phi}{dT''} = \alpha\left(\frac{\partial\Phi}{\partial T} + v\Phi_{|1}\right),\quad \frac{d\Phi}{dT'} = \alpha'\left(\frac{\partial\Phi}{\partial T} + v'\Phi_{|1}\right).$$

Since the quantities of the Schwarzschild geometry do not explicitly depend on the time the first terms vanish in the brackets. If one uses the auxiliary relations

$$d\alpha v = \alpha^3 dv,\quad d\alpha = \alpha^3 v dv,\quad dv' = \frac{\alpha^2}{\alpha'^2} dv\,, \tag{2.14}$$

and furthermore

$$v_{|1} = \frac{\partial v}{\alpha\partial r} = \frac{1}{\alpha\rho},\quad \rho = \sqrt{\frac{2r^3}{M}}\ ,$$

ρ being the radius of curvature of the Schwarzschild parabola, we obtain for the forces acting on the unit mass

$$G_{1''} = \alpha^2 v\frac{1}{\rho},\quad G_{1'} = \alpha\alpha' v'\frac{1}{\rho}\ . \tag{2.15}$$

With the auxiliary formulae (1.14) one gets

$$\frac{d\alpha}{dT''} = \alpha v G_{1''},\quad \frac{d\alpha'}{dT'} = \alpha' v' G_{1'}\,, \tag{2.16}$$

or

$$\frac{dm}{dT} = mvG_{1''} = vK_{1''},\quad \frac{dm'}{dT} = m'v'G_{1'} = v'K_{1'}\ . \tag{2.17}$$

The last two equations describe the work which is done on accelerating objects per time unit.

Following the idea of Einstein's elevator we first refer to the gravity-free space. If we accelerate the elevator downwards by external forces, objects in the elevator are accelerated towards the ceiling of the elevator with the counter force -G because of their inertia. If we interpret the external force as the gravitational force, it acts likewise on the objects inside the elevator and nullifies the counterforce. The objects in the elevator are hovering during the free fall. We will show that this is equally valid for both cases addressed.

We know from the special theory of relativity that the accelerations between relatively moving systems do not transform as vectors. In the context of the gravitation theory this is expressed by the inhomogeneous transformation law of the Ricci-rotation coefficients. To incorporate gravity into the problem in the above relations the partial derivatives have to be replaced by covariant ones. We arrive at the Ricci-rotation coefficients and the forces contained in them.

For our considerations it is sufficient to investigate the radial and time-like part of the metric. With

$$E_{nm}{}^{s} = h_{nm}E^{s} - h_{n}{}^{s}E_{m} \qquad (2.18)$$

one has for a four-vector

$$\Phi_{m||n} = \Phi_{m|n} - E_{nm}{}^{s}\Phi_{s}\,.$$

Therein is

$$h_{mn} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \qquad (2.19)$$

the unit matrix in two-dimensional [1,4]-subspace which we have considered and

$$E_{m} = \left\{ \alpha v \frac{1}{\rho}, 0, 0, 0 \right\} \qquad (2.20)$$

the force of gravity. Therein v is identical with the velocity of a freely falling observer, and is negative. Therefore E is directed inside. α is the Lorentz factor associated with v.

First, we transform with (2.4) the gravitational field strength (2.18) into the accelerated systems and we get

$$E_{n''m''}{}^{s''} = h_{n''m''}E^{s''} - h_{n''}{}^{s''}E_{m''}, \quad E_{n'm'}{}^{s'} = h_{n'm'}E^{s'} - h_{n'}{}^{s'}E_{m'}\ , \qquad (2.21)$$

$$E_{m''} = \left\{ \alpha E_{1}, 0, 0, -i\alpha v E_{1} \right\}, \quad E_{m'} = \left\{ \alpha' E_{1}, 0, 0, -i\alpha' v' E_{1} \right\}. \qquad (2.22)$$

However, we note that the quantities (2.22) are the field strengths of the static system measured by an accelerated observer, but not the forces that act in this system. From the Lorentz invariance of the covariant derivatives

$$L_{n'm'}^{n\,m}\Phi_{n||m} = \Phi_{m'|n'} - \left[E_{n'm'}{}^{s'} + L_{s}^{s'} L_{m'|n'}^{s} \right]\Phi_{s'} \qquad (2.23)$$

the following relations

$$''E_{n''m''}{}^{s''} = E_{n''m''}{}^{s''} + L_s^{s''} L^s_{m''|n''}, \quad 'E_{n'm'}{}^{s'} = E_{n'm'}{}^{s'} + L_s^{s'} L^s_{m'|n'} \ . \tag{2.24}$$

result.

They express the inhomogeneous transformation law of the Ricci-rotation coefficients. The new variables describe the forces acting on the two types of observer. First, we exploit the Lorentz terms

$$\begin{gathered} L_s^{s''} L^s_{m''|n''} = h_{n''}{}^{s''} G_{m''} - h_{m''n''} G^{s''}, \quad L_s^{s'} L^s_{m'|n'} = h_{n'}{}^{s'} G_{m'} - h_{m'n'} G^{s'} \\ G_{m''} = \left\{ \alpha^2 v \frac{1}{\rho}, 0, 0, -i\alpha^2 \frac{1}{\rho} \right\}, \quad G_{m'} = \left\{ \alpha\alpha' v' \frac{1}{\rho}, 0, 0, -i\alpha\alpha' \frac{1}{\rho} \right\} . \end{gathered} \tag{2.25}$$

We have gotten to know the first components of G with (2.15). The relevant fourth components arise from the covariant approach (2.24). The quantities are connected by the Lorentz transformation

$$G_{m''} = L^{m'}_{m''} G_{m'} \ , \tag{2.26}$$

which subsequently justifies the choice of their name. Therewith one can calculate the effective field strengths

$$\begin{gathered} ''E_{n''m'''}{}^{s''} = h_{n''m''}\, ''E^{s''} - h_{n''}{}^{s''}\, ''E_{m''}, \quad 'E_{n'm'}{}^{s'} = h_{n'm'}\, 'E^{s'} - h_{n'}{}^{s'}\, 'E_{m'} \\ ''E_{m''} = \left\{ 0, 0, 0, -\frac{i}{\rho} \right\}, \quad 'E_{m'} = \left\{ \alpha_0 v_0 \frac{1}{\rho}, 0, 0, -i\alpha_0 \frac{1}{\rho} \right\} \end{gathered} \tag{2.27}$$

which act in the observer systems.

The equation for the geodesics of an observer who comes from infinity, we write according to (2.23) and (2.24) as

$$''u_{m''||n''}\, ''u^{n''} = ''u_{m''|n''}\, ''u^{n''} - ''E_{n''m''}{}^{s''}\, ''u_{s''}\, ''u^{n''} = 0, \quad ''u_{m''} = \{0, 0, 0, 1\} \ . \tag{2.28}$$

It includes the components of the four-velocity of a freely falling observer in his own frame of reference. Since the first component of $''E$ vanishes, it is evident that observers hover in Einstein's elevator which comes from infinity.

The relation (2.28) is trivially satisfied. More insight is obtained if one measures the 4-velocity of the free-falling observer in a static system. With (2.4) one has

$$''u_m = \{ -i\alpha v, 0, 0, \alpha \} \ . \tag{2.29}$$

This we insert into

$$''u_{m||n}\, ''u^n = 0 \ , \tag{2.30}$$

a relation which is equivalent to (2.28), and we put it into a more stringent covariant form[1]

$$''m^m\, ''u_{m||n}\, ''u^n = 0 \ . \tag{2.31}$$

Therein is

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[1] Eq. (2.30) leads with $''u_{1||n}\, ''u^n = 0$ to the Minkowski force, which differs from the Lorentz force by the Lorentz factor. Accurate measurements of the electron have shown that the Lorentz force is the correct definition of the force in the theory of moving media in electrodynamics. We believe that one has to use the Lorentz force in the theory of gravity as well.

$$''m_{m''} = \{1,0,0,0\}, \quad ''m_m = \{\alpha,0,0,i\alpha v\} \tag{2.32}$$

the vector in the 1-direction of the local pseudorotated 2-bein. Expanding (2.31)

$$''m^1 \, ''u_{1|n} \, ''u^n + ''m^4 \, ''u_{4|n} \, ''u^n - E_{nm}{}^s \, ''u^n \, ''u_s \, ''m^m = 0 \ , \tag{2.33}$$

one gets by use of (2.29), (2.32), and due to $x^{1''} = \text{const.}$ with

$$''u^n \partial_n = ''u^{n''} \partial_{n''} = \frac{d}{idT''}$$

the relation

$$-\frac{1}{\alpha}\frac{d\alpha v}{dT''} - E_{4''1''}{}^{4''} = 0 \, .$$

According to (2.21) and (2.22) one finally arrives at

$$-\frac{1}{\alpha}\frac{d\alpha v}{dT''} + E_{1''} = 0 \, . \tag{2.34}$$

The first term in this relation is the counter force which tries to pull the objects in the elevator to the ceiling. With (2.7) and (2.15) one has

$$-G_{1''} + E_{1''} = 0 \, , \tag{2.35}$$

a relation which is satisfied with (2.15) and (2.22), first equation. The counter force is nullified by the gravity.

3. FREE FALL FROM AN ARBITRARY POSITION

From (2.7) one could easily derive the equation of motion for the observer B'. However, this simple method would conceal the interaction of forces. The relation (2.28) may be written as

$$''m^{m'} \, ''u_{m'||n'} \, ''u^{n'} = ''m^{m'} \, ''u_{m'|n'} \, ''u^{n'} - 'E_{n'm'}{}^{s'} \, ''u_{s'} \, ''u^{n'} \, ''m^{m'} = 0 \, , \tag{3.1}$$

as can be shown with a Lorentz transformation of the Ricci-rotation coefficients

$$'E_{n'm'}{}^{s'} = ''E_{n'm'}{}^{s'} + L^{s'}_{s''} L^{s''}_{m'|n'} \, . \tag{3.2}$$

Since the second term in this equation vanishes according to (2.4) only

$$'E_{n'm'}{}^{s'} = ''E_{n'm'}{}^{s'} \tag{3.3}$$

remains.

The

$$''m^{m'} = \{\alpha_0, 0, 0, i\alpha_0 v_0\}, \quad ''u^{n'} = \{-i\alpha_0 v_0, 0, 0, \alpha_0\} \tag{3.4}$$

are the unit vectors in the local 1- and 4-directions of system B", as measured in the system B'. Decomposing (3.1) in compliance with (3.4), second equation, one first has

$$''m^{m'}\,''u_{m'||n'}\,''u^{n'} = -i\alpha_0 v_0\,''m^{m'}\,''u_{m'||n'}\,'m^{n'} + \alpha_0\,''m^{m'}\,'u_{m'||n'}\,'u^{n'} = 0\,. \qquad (3.5)$$

In the first term on the right side the partial derivatives vanish due to the constancy of the parameters contained in it. Using (3.3), it becomes

$$i\alpha_0 v_0\,''E_{n'm'}{}^{s'}\,'m^{n'}\,''m^{m'}\,''u_{s'} = i\alpha_0 v_0\,''E_{n''m''}{}^{s'}\,'m^{n''}\,''m^{m''}\,''u_{s''} = i\alpha_0 v_0\,''E_{n''1''}{}^{4''}\,'m^{n''}\,. \qquad (3.6)$$

With

$$'m^{n''} = \{\alpha_0, 0, 0, -i\alpha_0 v_0\} \qquad (3.7)$$

and with (2.27) one has in (3.6)

$$i\alpha_0 v_0 \cdot \alpha_0\,''E_{1''1''}{}^{4''} = \alpha_0 \cdot i\alpha_0 v_0 \cdot \frac{i}{\rho} = \alpha_0\,'E_{1'}\,.$$

Inserting into (3.5) we obtain

$$''m^{m'}\,'u_{m'||n'}\,'u^{n'} - {}'E_{1'} = 0\,. \qquad (3.8)$$

If we rewrite the first term for a better understanding, we have

$$\frac{1}{\alpha_0}\frac{D\alpha_0 v_0}{DT'} = -{}'E_{1'}\,, \qquad (3.9)$$

the covariant equation of motion for the auxiliary system moving with the constant velocity v_0. It is evident that $''m^{m'}\,'u_{m'||n'}\,'u^{n'}$ is not the equation of a geodesic. To realize such a constant motion in the Schwarzschild field, one would need an external force 'E, which forces the comoving observer to his path.

To recall the problem of Einstein's elevator we start again from the equation of motion of a freely falling system which comes from infinity, namely in the form (2.31). This time we decompose with a Lorentz transformation

$$''m^m = \alpha_0\,'m^m + i\alpha_0 v_0\,'u^m, \quad ''u_m = -i\alpha_0 v_0\,'m_m + \alpha_0\,'u_m\,. \qquad (3.10)$$

This first results in

$$-i\alpha_0 v_0\left[\alpha_0\,'m^m + i\alpha_0 v_0\,'u^m\right]'m_{m||n}\,''u^n + \alpha_0\left[\alpha_0\,'m^m + i\alpha_0 v_0\,'u^m\right]'u_{m||n}\,''u^n = 0\,.$$

With (2.18) and with

$$'m_m = \{\alpha', 0, 0, i\alpha' v'\}, \quad 'u_m = \{-i\alpha' v', 0, 0, \alpha'\} \qquad (3.11)$$

the expressions

$$'m^m\,'m_{m||n} = 0, \quad 'u^m\,'u_{m||n} = 0 \qquad (3.12)$$

vanish. Furthermore, we use the orthogonality of the vectors m and u

$$'u^m\,'m_{m||n} = -{}'m^m\,'u_{m||n}\,, \qquad (3.13)$$

so that we obtain as an intermediate result

$$'m^m\,'u_{m||n}\,''u^n = 0\,. \qquad (3.14)$$

This is expanded with (3.10), second equation, into

$$-i\alpha_0 v_0\,'m^m\,'u_{m||n}\,'m^n + \alpha_0\,'m^m\,'u_{m||n}\,'u^n = 0\,. \qquad (3.15)$$

Due to the Lorentz invariance of the covariant derivative one has in the first term also

$$'m^{m'}\,'u_{m'||n'}\,'m^{n'} = -'E_{n'm'}{}^{s'}\,'u_{s'}\,'m^{m'}\,'m^{n'} = -'E_{4'} = -i\alpha_0 \frac{1}{\rho}\,.$$

With (3.3) one lastly obtains

$$'m^{m}\,'u_{m||n}\,'u^{n} - ''E_{1'} = 0, \quad ''E_{1'} = \alpha_0 v_0 \frac{1}{\rho}\,. \tag{3.16}$$

Expanding the covariant derivative one has

$$'m^{m}\,'u_{m|n}\,'u^{n} - E_{nm}{}^{s}\,'u_{s}\,'u^{n}\,'m^{m}\ ,$$

whereby the second term results in

$$-E_{n'm'}{}^{s'}\,'u_{s'}\,'u^{n'}\,'m^{m'} = -E_{4'1'}{}^{4'} = E_{1'}\,.$$

while the first term is calculated with (3.11). Thus, the equation of motion for an observer freely falling away from r_0 can be written as

$$-\frac{1}{\alpha'}\frac{d\alpha' v'}{dT'} + E_{1'} - ''E_{1'} = 0\,. \tag{3.17}$$

Thus, the effective force of gravity acting in the system B' is the difference of the effective gravity acting in the static system B, as it can be measured in the system B, and the effective gravity in the system B'', as it can be measured in system B'.

It turns out that the forces acting on a system freely falling from r_0 can be calculated only by indirection via the system falling in from the infinite. The reason is that the velocity v' can only be derived by means of (2.1) from the speed of the system B''. The non-covariant behavior of the acceleration occurs in the context of the gravitation theory as the inhomogeneous transformation law of the Ricci-rotation coefficients.

The two E-terms in (3.17) lead to Einstein's composition law of velocities

$$\alpha' \alpha v \frac{1}{\rho} - \alpha_0 v_0 \frac{1}{\rho} = \alpha\alpha' v' \frac{1}{\rho}\,.$$

The first term in (3.17) we have previously calculated with (2.15). Thus, we recognize that the counter force

$$-G_{1'} = ''E_{1'} - E_{1'} \tag{3.18}$$

is nullified by the effective gravity in the system B'. Objects in an elevator freely falling from r_0 hover. On should compare this with the relation (2.35).

If one performs the Newtonian approximation in (3.17) by putting the Lorentz factors 1 and if one considers that the proper time coincides with the absolute time T, one has for the counter force in the elevator

$$-\frac{dv'}{dT} = -\frac{M}{r^2}\left(1 - \sqrt{\frac{r}{r_0}}\right). \tag{3.19}$$

One makes out that the counter force vanishes for $r = r_0$, viz if the elevator is still at rest. If the elevator is falling in from the infinite one has $v' = v$. In this case the root in the brackets vanishes for $r_0 \to \infty$. Therefore remains

$$m\frac{d(-v)}{dT} = -\frac{mM}{r^2}, \tag{3.20}$$

the Newtonian formula for the force in geometric units.

Lemaître [4,5] has introduced a co-ordinate system that brings the metric into a time-dependent form. This co-ordinate system follows the observer who comes from the infinite. The co-ordinate transformation may be written as

$$\begin{array}{llll} \Lambda^{1}_{1''} = -v, & \Lambda^{1}_{4''} = -iv, & \Lambda^{4}_{1''} = -i\alpha^2 v^2, & \Lambda^{4}_{4''} = \alpha^2 \\ \Lambda^{1''}_{1} = -\dfrac{\alpha^2}{v}, & \Lambda^{1''}_{4} = -i, & \Lambda^{4''}_{1} = -i\alpha^2 v, & \Lambda^{4''}_{4} = 1 \end{array}. \tag{3.21}$$

Here the indices are co-ordinate indices. We calculate a 2-bein adapted to the co-ordinate system in the 2-dimensional subspace which we have considered. If the $\overset{m}{e}_i$ are the Schwarzschild-standard vectors one obtains with

$$\overset{m''}{e}_{i''} = L^{m''}_{m} \overset{m}{e}_{i} \Lambda^{i}_{i''} \tag{3.22}$$

the values

$$\overset{1''}{e}_{1''} = -v, \quad \overset{4''}{e}_{4''} = 1, \quad \underset{1''}{e}^{1''} = -\frac{1}{v}, \quad \underset{4''}{e}^{4''} = 1. \tag{3.23}$$

With (3.23) one can immediately write down the metric in Lemaître co-ordinates. Since the redshift factor in this system is equal to 1, no gravitational field strength is contained in the Ricci-rotation coefficients. A freely falling observer is weightless. From the 1-components of the bein vectors the radial tidal force can be determined.

Lemaître-like co-ordinates can also be found for an observer who falls away from r_0. Analogous to (3.22) the corresponding Λs can be calculated with

$$\Lambda^{i'}_{i} = \underset{m'}{e}^{i'} L^{m'}_{m} \overset{m}{e}_{i} \tag{3.24}$$

The relation, however, contains more unknown quantities than equations. Thus, we need additional conditions:

1. The transformation Λ must be holonomic.
2. It must be orthogonal.
3. It should not explicitly depend on the Schwarzschild-standard time co-ordinate.
4. Λ has to coincide with the Lemaître transformation (3.21) for $r_0 \to \infty$.
5. The new 2-bein vectors must be orthogonal, they must not contain 1' and 4' mixed indices, so that no cross terms emerge in the metric.

To comply with condition 1 $\Lambda^{i'}_{[i|k]}=0$ must hold, which leads immediately to $\Lambda^{i'}_{4|1}-\Lambda^{i'}_{1|4}=0$ and reduces to $\Lambda^{1'}_{4|1}=0$ and $\Lambda^{4'}_{4|1}=0$ due to condition 3. This means that $\Lambda^{4'}_{4}$ must be a real constant that one can put to 1 and $\Lambda^{1'}_{4}$ should be an imaginary constant which we put i, both in accordance with the fourth condition. Since both the Lorentz transformation and the static bein vectors

$$\overset{1}{e}_1=\alpha, \quad \overset{4}{e}_4=\frac{1}{\alpha}, \quad \underset{1}{e}^1=\frac{1}{\alpha}, \quad \underset{4}{e}^4=\alpha \tag{3.25}$$

are known, it is easy to calculate the remaining components of $\Lambda^{i'}_{i}$ with the help of condition 5, and at the same time the new 2-bein. With the conditions 2 and 5 one gets the reciprocal values of the two quantities. They are

$$\begin{array}{llll} \Lambda^{1}_{1'}=-\dfrac{\alpha'^2}{\alpha^2}v', & \Lambda^{1}_{4'}=-i\dfrac{\alpha'^2}{\alpha^2}v', & \Lambda^{4}_{1'}=-i\alpha'^2 v'^2, & \Lambda^{4}_{4'}=\alpha'^2 \\ \Lambda^{1'}_{1}=-\dfrac{\alpha^2}{v'}, & \Lambda^{1'}_{4}=-i, & \Lambda^{4'}_{1}=-i\alpha^2 v', & \Lambda^{4'}_{4}=1 \end{array}, \tag{3.26}$$

$$\overset{1'}{e}_{1'}=-\frac{\alpha'}{\alpha}v', \quad \overset{4'}{e}_{4'}=\frac{\alpha'}{\alpha}, \quad \underset{1'}{e}^{1'}=-\frac{\alpha}{\alpha'}\frac{1}{v'}, \quad \underset{4'}{e}^{4'}=\frac{\alpha}{\alpha'}. \tag{3.27}$$

This leads to the metric

$$ds^2=\frac{\alpha'^2}{\alpha^2}\left(v'^2 dr'^2-dt'^2\right). \tag{3.28}$$

For $v_0=0$ and thus for $\alpha'=\alpha$ one immediately obtains the Lemaître metric for freely falling observers coming from the infinite

$$ds^2=v^2 dr''^2-dt''^2. \tag{3.29}$$

It can be seen that only for observers who come from infinity, the co-ordinate time t'' coincides with the proper time T''. The metric (3.28) can alternatively be written as

$$ds^2=\alpha_0^2\left[\left(v-v_0\right)^2 dr'^2-\left(1-vv_0\right)^2 dt'^2\right]. \tag{3.30}$$

The Ricci-rotation coefficients calculated from the 2-bein (3.27) are merely the already-known primed quantities of (2.27).

4. SUMMARY

By elementary operations like Lorentz transformations and the Einstein composition law of velocities we have derived the velocities and accelerations of freely falling objects, falling in from arbitrary positions in the Schwarzschild field. We have also derived the forces acting on objects in Einstein's elevator and we have shown that these objects are hovering.

5. REFERENCES

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