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PENROSE METRIC AND BLACK HOLES

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Abstract: The Penrose metric is investigated, and it is shown that it cannot describe the inner region of the Schwarzschild model. It cannot represent a black hole.

1. INTRODUCTION

In a summary paper [1] Penrose[1] presented a coordinate transformation that modifies the standard Schwarzschild metric allowing it to describe the inner region of the Schwarzschild model. In further papers [2-3], he refined this point of view. We demonstrate that this is not possible. We calculate the field strengths based on the metrical coefficients of the Penrose metric and show that the gravitational force tends to infinity when an observer approaches the event horizon at $r = 2M$. Therefore, a penetration of the event horizon is not possible, even though the radial coordinate r of Penrose ranges from $r = 0$ to $r = \infty$. This prevents the formation of a black hole.

2. THE PENROSE METRIC

Penrose derived a new metric from the standard Schwarzschild metric using the coordinate transformation

$$t = t' + r + 2M\ln(r - 2M), \quad r = r' \,, \tag{2.1}$$

where $\{r,t\}$ are the new coordinates, and $\{r',t'\}$ are the standard Schwarzschild coordinates. He noted that these had already been used by Eddington [4] and Finkelstein [5]. However, we must consider that the Penrose coordinates differ slightly from these[2]. The coordinate t is referred to as the advanced time coordinate.

By differentiating (2.1), we get

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[1] Penrose has received several prizes and awards, including the 2020 Nobel Prize in Physics for the discovery that black hole formation is a robust prediction of the general theory of relativity.

[2] Finkelstein: $t = t' + 2M\ln(r - 2M)$, Eddington: $t = t' - 2M\ln(r - M)$.

$$dt = dt' + \alpha^2 dr, \quad a = \sqrt{1-2M/r}, \quad \alpha = 1/a\,, \tag{2.2}$$

and, finally, the Penrose metric

$$ds^2 = 2drdt + r^2 d\Omega^2 - a^2 dt^2\,. \tag{2.3}$$

Here, Ω is the solid angle of the spherical Schwarzschild coordinate system. The form of the metric does not impose any condition on the variable r. This gives the impression that the metric can also describe the inner region of the Schwarzschild model. We see that the Penrose coordinate system is oblique-angled, i.e., the Penrose coordinates r and t are not orthogonal.

From the line element (2.3), we read the metrical coefficients using the original Minkowski notation $x^4 = i(c)t$

$$\begin{array}{lll} g_{11} = 0, & g_{14} = -i, & g_{44} = a^2 \\ g^{11} = a^2, & g^{14} = \ \ i, & g^{14} = 0 \end{array}\,. \tag{2.4}$$

From these, we extract the tetrads

$$\begin{array}{lll} \overset{1}{e}_1 = \alpha, & \overset{4}{e}_1 = -i\alpha, & \overset{4}{e}_4 = \ a \\ \underset{1}{e}^1 = a, & \underset{1}{e}^4 = i\alpha, & \underset{4}{e}^4 = \alpha \end{array}\,. \tag{2.5}$$

They represent rods and clocks. They connect coordinate objects Φ_i with measured values by $\Phi_m = \underset{m}{e}^i \Phi_i$. The field strengths for the Penrose metric are calculated using the Ricci-rotation coefficients:

$$A_{mn}{}^s = \overset{s}{e}_j \underset{[n|m]}{e}^j + g^{sr} g_{nt} \overset{t}{e}_j \underset{[m|r]}{e}^j - g^{sr} g_{mt} \overset{t}{e}_j \underset{[r|n]}{e}^j = \overset{s}{e}_{j'} \underset{[n|m]}{e}^{j'} + g^{sr} g_{nt} \overset{t}{e}_{j'} \underset{[m|r]}{e}^{j'} - g^{sr} g_{mt} \overset{t}{e}_{j'} \underset{[r|n]}{e}^{j'} = \ldots.\ . \tag{2.6}$$

The Ricci-rotation coefficients are invariant under coordinate transformations if the coordinate transformation is holonomic. Thus, we do not expect new coordinates to change the physical content of a model. Invariance under coordinate transformations is one of the basic principles of the theory of relativity and is equally indispensable for all physics. Mathematical details concerning the coordinate invariance of (2.6) can be found in the Mathematical Appendix.

Remembering that is $g_{mn} = \delta_{mn}$ in the tetrad formalism, we calculate a typical element of the Ricci-rotation coefficients using the Penrose metric and definitions (2.2):

$$A_{41}{}^4 = -\overset{4}{e}_4 \underset{4|1}{e}^4 = -a\left(\frac{1}{a}\right)_{|1} = \alpha \frac{M}{r^2}\,.$$

Finally, we have the force of gravity

$$E = -\frac{1}{\sqrt{1-2M/r}} \frac{M}{r^2}\,. \tag{2.7}$$

It is not very surprising that we get the same expression for gravity that we calculate with the standard Schwarzschild coordinates.

Evidently, for $r \to 2M$, we obtain $E \to \infty$. This means that the event horizon at $r = 2M$ is a physical singularity, while the standard Schwarzschild metric has a coordinate singularity at $r = 2M$. The latter can be removed by a suitable coordinate transformation.

The fact that the variable r in the Penrose metric is regular everywhere does not prevent the gravitational force from diverging at the event horizon (Fig.1).

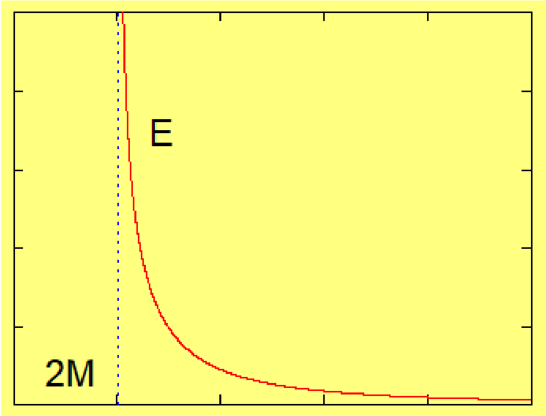

Fig. 1. The gravitational force

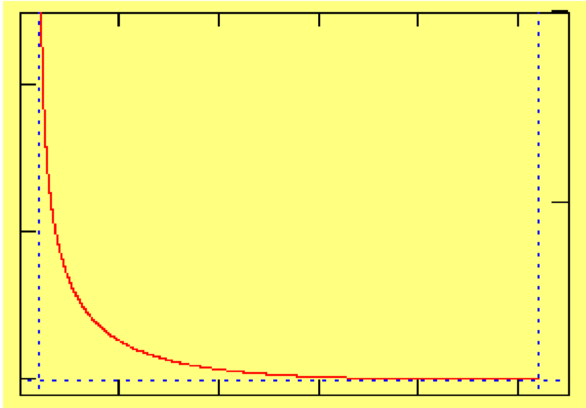

Fig. 2. Proper time of free fall

There are other coordinate systems that are regular at $r = 2M$. These include the coordinate systems of Einstein and Rosen and isotropic coordinates. They have the additional property that neither of them can describe the region $r < 2M$ of the Schwarzschild model. We have outlined the possibility of using new coordinates in our paper [6].

The literature often overlooks the existence of a coordinate system that covers the whole space and is regular everywhere: the Cartesian coordinates $\{R, r\}$ of the flat embedding space of the Schwarzschild geometry. R is the coordinate of the extradimension, orthogonal to r. By implementing the Schwarzschild model in this space using the equation of Flamm's paraboloid

$$R^2 = 8M(r - 2M),$$

we must accept that the range of r is restricted to $r = \{2M, \infty\}$, unless one allows the variable R to become imaginary. The equation above defines the geometry and leaves no need to search for a coordinate system that opens access to other regions in space. Moreover, the proper time of a free-falling observer approaching the event horizon diverges, as shown in Fig. 2. We have calculated this in our paper [7] and have explored the problem with various methods in [8]. Additionally, every observer in free fall asymptotically approaches the event horizon at the speed of light. We show this in Fig. 3. We addressed this problem in a talk in Berlin a few years ago, and a detailed discussion of this question can be found in the English translation of the paper in [9]. All in all, the event horizon marks the end of the exterior Schwarzschild geometry.

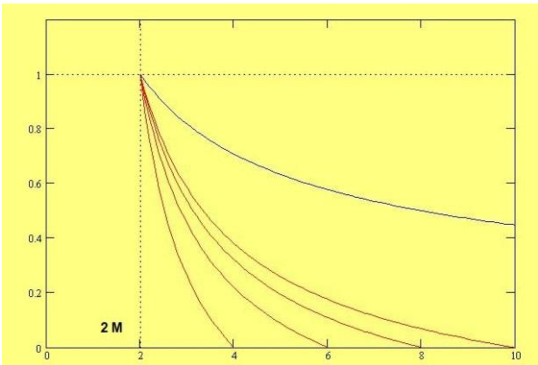

Fig. 3. Free fall from arbitrary positions

Penrose wrote in his paper [1]:

"The observer … crosses freely from the $r > 2M$ *region into the* $0 < r < 2M$ *region. He encounters* $r = 2M$ *at a perfectly finite time, according to his own local clock, and he experiences nothing special at this point. … Let us consider another observer, however, who is situated far from the star. As we trace the light rays from his eye, back into the past towards the star, we find that they cannot cross into the* $r < 2M$ *region after the star has collapsed through. … No matter how long the external observer waits, he can always (in principle) still see the surface of the star as it was just before it plunged through the Schwarzschild radius. In practice, however, he would soon see nothing of the star's surface – only a 'black hole' – since the observed intensity would die off exponentially, owing to an infinite red shift."*

These considerations by Penrose do not align with the features of the exterior Schwarzschild solution which one obtains by careful calculations. In his textbook, Mitra [10] discussed the problems of the inner region of the Schwarzschild geometry in detail. Since the exterior Schwarzschild geometry ends at the event horizon, the question of how to fill the hole at the waist of Flamm's paraboloid arises. The answer was given by Schwarzschild in the year 1916. The interior Schwarzschild solution matches smoothly with the exterior Schwarzschild solution, and the first and second linking conditions are satisfied. The interior solution provides pressure and mass density; there is no reason to assume that our universe could be full of holes.

Penrose also described what he believed was going on in the inner region of the Schwarzschild model:

"As r *decreases, the space-time curvature mounts (in proportion to* r^{-3}*), becoming theoretically infinite at* $r = 0$*. … Thus, the true space-time singularity, resulting from a spherical symmetric collapse, is located not at* $r = 2M$*, but at* $r = 0$*. Although the hypersurface* $r = 2M$ *has, in the past, itself been frequently referred as the 'Schwarzschild singularity', this is really a misunderstanding terminology… ."*

It is hard to understand that mass could be concentrated in a single point with infinite mass density and infinitely high curvature of space. In contrast, the collapse of the Schwarzschild interior [11] shows that the stellar object can only asymptotically contract to an object having a minimum radius of $r_H = 2.25M$. At this location, the pressure at the center of the star would be infinite. Moreover, the surface of the star needs an infinitely long proper time to approach this inner horizon r_H. After some time, the collapse slows down to a quasi-static state of the object. Mitra called such a star an ECO (Eternally Collapsing Object). Such an object could be in the centers of galaxies, possessing ultra-high density, and not emitting light due to the strong gravitational field; it would appear to be black.

Fig. 4. shows the collapse of the interior Schwarzschild solution. This solution is geometrically represented by the cap of a sphere. This cap slides down the Schwarzschild parabola in such a way that the tangents of both the interior and exterior solutions coincide. The exterior solution remains unaffected by the collapse, as per the Birkhoff theorem.

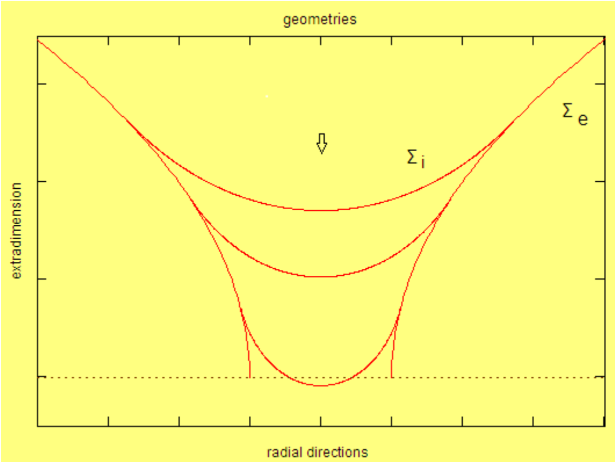

Fig. 4. Collapse of the interior Schwarzschild solution

A further remark of Penrose is:

"Let us imagine a situation in which the collapse of a spherically symmetrical (nonrotating) star takes place and continues until the surface of the star approaches the Schwarzschild radius. So long as the star remains spherically symmetrical, its external field remains that given by the Schwarzschild metric. … It would seem that the surface of the star can never cross to within the $r = 2M$ *region. However, this is misleading. For suppose an observer were to follow the surface of the star in a rocket ship, down to* $r = 2M$*. He would find (assuming that collapse does not differ significantly from free fall) that the total proper time that he would experience as elapsing, as he finds his way down to* $r = 2M$*, is in fact finite."*

Reading this, we get the impression that according to Penrose, the collapse of a star is solely a matter of the exterior Schwarzschild solution. However, we must bear in mind that any particle on the surface of a collapsing star follows the laws of an interior solution.

In paper [3], Penrose noted that even deviations from spherical symmetry cannot prevent spacetime singularities from arising. Concerning collapse, he refers to the Oppenheimer and Snyder model. This model holds strong historical importance but cannot be used to explain a physically model relevant collapse, as it assumes pressureless Nature. A direct reference to collapsing models is absent in Penrose's work.

3. CONLUSIONS

The Penrose metric was derived from the standard Schwarzschild metric through a coordinate transformation with an advanced time coordinate. We demonstrated that this transformation – as well as any other coordinate transformation – does not alter the geometry, i.e., the physical content of the model. The coordinate singularity at $r = 2M$ can be removed with a suitable coordinate transformation, but the physical singularity at $r = 2M$ remains.

This invalidates all of Penrose's speculations about singularities. The event horizon cannot be crossed, and no singularity emerges at $r = 0$, whether naked or dressed. There is also no room for trapped surfaces. The complete Schwarzschild theory, consisting of the interior and exterior Schwarzschild solutions, is regular everywhere, and the force of gravity is zero at the center of the star. When a stellar object collapses, it becomes an ECO.

4. MATHEMATICAL APPENDIX

In this appendix, we demonstrate the invariance of the Ricci-rotation coefficients, which contain the physical quantities of the model, under a holonomic coordinate transformation. To achieve this, we use the transformation

$$\overset{m}{e}_{i} = \overset{m}{e}_{i'} \Lambda^{i'}_{i}, \quad \underset{m}{e}^{i} = \underset{m}{e}^{i'} \Lambda^{i}_{i'} .$$

Here, the Λ represent the transformation coefficients for the coordinates. The Ricci-rotation coefficients, using the coordinate system i', are

$$A_{mn}{}^{s} = \overset{s}{e}_{j'} \underset{[n|m]}{e}^{j'} + g^{sr} g_{nt} \overset{t}{e}_{j'} \underset{[m|r]}{e}^{j'} - g^{sr} g_{mt} \overset{t}{e}_{j'} \underset{[r|n]}{e}^{j'} .$$

Applying the above coordinate transformation, we get

$$\begin{aligned} A_{mn}{}^{s} &= \overset{s}{e}_{j'} \left[\Lambda^{j'}_{j} \underset{[n|}{e}^{j} \right]_{|m]} + g^{sr} g_{nt} \left[\Lambda^{j'}_{j} \underset{[m}{e}^{j} \right]_{|r]} - g^{sr} g_{mt} \left[\Lambda^{j'}_{j} \underset{[r}{e}^{j} \right]_{|n]} \\ &= \overset{s}{e}_{j} \underset{[n|m]}{e}^{j} + g^{sr} g_{nt} \overset{t}{e}_{j} \underset{[m|r]}{e}^{j} - g^{sr} g_{mt} \overset{t}{e}_{j} \underset{[r|n]}{e}^{j} + \underset{m}{e}^{i} \underset{n}{e}^{j} \overset{s}{e}_{j'} \Lambda^{j'}_{[j|i]} + g^{sr} g_{nt} \overset{t}{e}_{j'} \underset{r}{e}^{i} \underset{m}{e}^{j} \Lambda^{j'}_{[j|i]} - g^{sr} g_{mt} \overset{t}{e}_{j'} \underset{r}{e}^{i} \underset{n}{e}^{j} \Lambda^{j'}_{[j|i]} \end{aligned} .$$

The condition for holonomy is

$$\Lambda^{j'}_{[j|i]} = x^{j'}{}_{|[j\,i]} = 0 .$$

Thus, the last three brackets in the above equation vanish. Applied to the Schwarzschild model, it is futile to search for a coordinate transformation that allows for the penetration of the event horizon. Furthermore, this proof is independent of the mathematical structure of any model.

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