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COMMENTS ON THE USE OF THE FLATNESS THEOREM BY MELIA

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Abstract: We refer to a recently published paper by Melia concerning the Local Flatness Theorem. We argue that Melia's claim that the timelike metrical coefficient $g_{tt}=1$ is closely connected with the equation of state $\mu_0+3p=0$ is based on a mathematical artifact. Melia's model is analyzed using coordinate independent methods and is compared with our Subluminal Model.

1. INTRODUCTION

Melia published a paper recently [1], wherein he derived the equation of state (EOS) of his R_h=ct model resorting to the Local Flatness Theorem. Performing a gauge transformation of the time variable in the comoving system of this expanding model, he putatively proved that the timelike metrical coefficient $g_{tt}=1$ is only satisfied if the EOS has the form $\mu_0+3p=0$. We recall that all cosmological models expanding in free fall have $g_{tt}=1$. We refer to the Friedman cosmos and the dS family. They have different EOS and no relations can be deduced from $g_{tt}=1$ and the specific EOS of the models.

In Sec 2. we show that the gauge transformation of the time variable results in a mathematical artifact. We study the field quantities and field equations of Melia's flat space ansatz and calculate the pressure and mass density of the cosmic fluid. In Sec. 3 we compare Melia's model with our Subluminal Model [2] and conclude that the R_h=ct model is not globally flat but locally flat, and thus identical with our globally curved Subluminal Model. In Sec. 4 we treat this in greater detail.

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2. THE GAUGE TRANSFORMATION

Melia starts his argument for the need of a gauge transformation with a transformation of the time variable

$$dt \rightarrow \sqrt{g_{tt}}dt, \quad L(t) = g_{tt}, \tag{2.1}$$

L being the lapse function of the metric

$$ds^2 = L(t)dt^2 - a(t)^2\left(dx^{1^2} + dx^{2^2} + dx^{3^2}\right). \tag{2.2}$$

The metric is written in a manner that suggests a flat expanding cosmological model, $a(t)$ being the time dependent scale factor. We analyze this ansatz in more detail.

For better processing and for comparing Melia's model with our Subluminal Model [2], we change the notation. We use the original Minkowski notation with $x^4 = ix^0 = i(c)t$, define $l^2 = L$, and relabel $a(t) \rightarrow \mathcal{K}(t)$ to avoid confusion with a quantity we use later on. Thus, we arrive at

$$ds^2 = \mathcal{K}^2\left(dx^{1^2} + dx^{2^2} + dx^{3^2}\right) + l^2 dx^{4^2}. \tag{2.3}$$

Melia evaluated the Christoffel symbols for the metric (2.2). Rewriting his results with the Minkowski notation and correcting for a missing factor of $1/2$ we get

$$\begin{aligned}
\Gamma^1_{14} &= \Gamma^2_{24} = \Gamma^3_{34} = \frac{1}{\mathcal{K}}\mathcal{K}_{|4} \\
\Gamma^4_{11} &= \Gamma^4_{22} = \Gamma^4_{33} = -\frac{1}{l^2}\mathcal{K}\mathcal{K}_{|4}. \\
\Gamma^4_{44} &= \frac{1}{l}l_{|4}
\end{aligned} \tag{2.4}$$

Here, all indices are coordinate indices.

We daresay the choice of coordinates is quite arbitrary, and a matter of convenience. It is evident that the components of the Christoffel symbols are not measured values, i.e., they are not quantities resulting from the use of rods and clocks. To get coordinate invariant quantities one has to use the Ricci-rotation coefficients. They are independent of the coordinate system in use and are calculated on the basis of tetrads, i.e., rods and clocks. Moreover, one has to use the Minkowski notation to obtain consistent results.

Eq. (C) in the appendix provides the relation between Ricci-rotation coefficients and Christoffel symbols. First, we read from (2.3) the tetrads

$$\begin{aligned}
&\overset{1}{e}_1 = \mathcal{K}, \quad \overset{2}{e}_2 = \mathcal{K}, \quad \overset{3}{e}_3 = \mathcal{K}, \quad \overset{4}{e}_4 = l \\
&\underset{1}{e}^1 = \frac{1}{\mathcal{K}}, \quad \underset{2}{e}^2 = \frac{1}{\mathcal{K}}, \quad \underset{3}{e}^3 = \frac{1}{\mathcal{K}}, \quad \underset{4}{e}^4 = \frac{1}{l}
\end{aligned} \tag{2.5}$$

and perform the following operations

$$
\begin{aligned}
A_{14}{}^{1} &= \underset{1}{e}{}^{1}\,\underset{4}{e}{}^{4}\,\overset{1}{e}{}_{1}\Gamma^{1}_{\underline{14}} = \underset{4}{e}{}^{4}\frac{1}{R}R_{|\underline{4}} = \frac{1}{R}R_{|4} \\
A_{11}{}^{4} &= \underset{1}{e}{}^{1}\,\underset{1}{e}{}^{1}\,\overset{4}{e}{}_{4}\Gamma^{4}_{\underline{11}} = \frac{1}{R^{2}}l\left(-\frac{1}{l^{2}}RR_{|\underline{4}}\right) = -\frac{1}{R}R_{|4} \\
A_{44}{}^{4} &= \underset{4}{e}{}^{4}\,\underset{4}{e}{}^{4}\,\overset{4}{e}{}_{1}\Gamma^{4}_{\underline{44}} + \overset{4}{e}{}_{4}\,\underset{4|4}{e}{}^{4} = \frac{1}{l}\frac{1}{l}l_{\underline{4}} + l\left(\frac{1}{l}\right)_{|4} = \frac{1}{l}l_{|4} - \frac{1}{l}l_{|4} = 0
\end{aligned}
\qquad (2.6)
$$

Here, we used the underbar for coordinate indices to distinguish them from tetrad indices and respected the relation $\partial_4 = \underset{4}{e}{}^{4}\partial_{\underline{4}} = \frac{1}{l}\partial_{\underline{4}}$. The relation $A_{44}{}^{4} = 0$ was to expect, because the Ricci-rotation coefficients have the symmetry property $A_{m(ns)} = 0$. It shows that the quantity $\frac{1}{l}l_{|\underline{4}}$ cannot be measured with rods and clocks in an orthogonal reference system and thus is a mathematical artifact. One cannot make physical conclusions from a gauge transformation of the time variable, using orthogonal *reference systems* instead of *coordinate systems*.

Mitra [6][7] analyzed the coordinate freedom of time in a most general way, starting with a metric for a spacetime:

$$ds^2 = e^{\lambda(r,t)}dr^2 + e^{\mu(r,t)}d\Omega^2 - e^{\nu(r,t)}dt^2,$$

where $e^{\mu/2}$ is the invariant circumference, also called the area coordinate, and Ω is the solid angle of a 2-sphere. If the parameters in this general metric remain to be time-dependent, they could describe an expanding universe. Then, r and t are interpreted as comoving coordinates. Demanding the cosmic fluid to be perfect, isotropic, and homogenous one can start with the stress-energy-momentum tensor $T_{11} = T_{22} = T_{33} = -p, T_{44} = \mu_0$. This choice constrains the parameters of the metric.

Mitra solves the field equations under the assumption that there is no radial heat flow ($T_{14} = 0$) but spatial homogeneity ($\mu_0 = \mu_0(t)$, $\mu_o' = 0$). Thus, the metric takes the following form

$$ds^2 = e^{\lambda}dr^2 + e^{\mu}d\Omega^2 - e^{\nu(t)}dt^2,$$

wherein e^{λ} and e^{μ} have the well-known FRW values. Invoking the coordinate freedom of time[1]

$$t \to t^{*}(t),$$

one can put $\nu = 0$ without loss of generality. No physical issue is connected with this restriction in contrast to Melia's statement. Finally, Mitra arrives at the standard FRW metric.

The fact that g_{44} is independent on r underlines that no acceleration (force on the unit mass) emerges. Moreover, Mitra differentiates the Friedman equation and obtains the acceleration equation of the FLRW model:

$$\frac{\ddot{R}}{R} = -\frac{4}{3}(\mu_0 + 3p).$$

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[1] For more details, please refer to the papers of Mitra [6][7].

Since no acceleration occurs in the employed system, one has $\ddot{\mathcal{K}} = 0$, and finally, the EOS

$$\mu_0 + 3p = 0,$$

valid for both Melia's and our Subluminal Model. Thus, the structure of the EOS is not connected to a gauge of the time coordinate but is a result of Einstein's field equations and the conservation law.

3. THE FIELD EQUATIONS

Having translated the description of the basic equations of the model into a coordinate independent form, we can proceed with the metric

$$ds^2 = \mathcal{K}^2\left(dx^{1^2} + dx^{2^2} + dx^{3^2}\right) + dx^{4^2} \tag{3.1}$$

and define a quantity

$$U_4 = A_{14}{}^{1} = A_{24}{}^{2} = A_{34}{}^{3} = \frac{1}{\mathcal{K}}\mathcal{K}_{|4}. \tag{3.2}$$

Furthermore, we introduce a constant $\mathcal{R}_0$ for later use and define

$$\mathcal{R} = \mathcal{K}\mathcal{R}_0. \tag{3.3}$$

Thus, the quantity U reads as

$$U_m = \left\{0,0,0,\frac{1}{\mathcal{R}}\mathcal{R}_{|4}\right\}. \tag{3.4}$$

In contrast to Melia's claim, the metric (2.2) and equally the metric (3.1) are not metrics of a flat space, because the Riemann does not vanish nor does the Ricci.

Using tetrad indices $m = 1,2,...,4$, the Ricci has the form

$$R_{mn} = A_{mn}{}^{s}{}_{|s} - A_{n|m} - A_{rm}{}^{s}A_{sn}{}^{r} + A_{mn}{}^{s}A_{s}, \quad A_n = A_{sn}{}^{s}. \tag{3.5}$$

One easily derives

$$R_{mn} = -\left[U^{s}{}_{|s} + 3U^{s}U_{s}\right]{}'g_{mn} - 3\left[U^{s}{}_{|s} + U^{s}U_{s}\right]u_m u_n. \tag{3.6}$$

Here, ${}'g_{mn} = \{1,1,1,0\}$ is the 3-dimensional tetrad metrical tensor and $u_m = \{0,0,0,1\}$ the timelike unit vector. Evaluating the Ricci scalar and Einstein tensor, we have

$$\begin{aligned} G_{11} = G_{22} = G_{33} &= 2U^{s}{}_{|s} + 3U^{s}U_{s} = \kappa p \\ G_{44} &= 3U^{s}U_{s} = -\kappa\mu_0 \end{aligned}. \tag{3.7}$$

We note that with these equations the possibilities of determining the quantity U are exhausted, because the line element describes the curvature of space, but cannot describe the change of curvature. The latter can be evaluated with the Bianchi identities. In their contracted version, they lead to the conservation law $T^{m}{}_{n||m} = 0$. Investigating this relation we get more information about the structure of the model

$$p_{|\alpha} = 0, \quad \mu_{0|4} = -3(p + \mu_0)U_4, \quad \alpha = 1,2,3 \tag{3.8}$$

The pressure of the cosmic fluid is spatially constant. To proceed, we refer to Eq. (3.4) and obtain

$$U^4{}_{|4} + U^4 U_4 = -\frac{1}{\mathcal{R}}\mathcal{R}_{|44} = \frac{1}{\mathcal{R}}\ddot{\mathcal{R}} = \frac{1}{\mathcal{K}}\ddot{\mathcal{K}}. \tag{3.9}$$

Following Melia we put

$$\ddot{\mathcal{K}} = 0, \tag{3.10}$$

i.e., the expansion of the universe does not accelerate. Melia has evidently accumulated an immense amount of astrophysical data acquired from observatories and has meticulously analyzed this data and provided diagrams in a series of papers. Recently, he and his coworkers reexamined [3] the data of the SDSS-IV Quasar Catalog with the Alcock-Paczyński effect. They modestly mentioned that the R_h=ct model is favored for explaining these data. Indeed, this is a strong argument for the prerequisite (3.10) for a cosmological model. We want to draw the reader's attention to a paper by Krasiński [4] ("*Cosmological models and misunderstanding about them.*"). He wrote: "*The accelerating expansion of the Universe is not an observed phenomenon, but an element of interpretation of observations, forced upon us by the R-W models*".

Accepting (3.10), Eq. (3.9) is reduced to

$$U^s{}_{|s} + U^s U_s = 0, \tag{3.11}$$

the Friedman equation in tensor form. It has the solution

$$U_4 = -\frac{i}{\mathcal{R}}, \quad \dot{\mathcal{R}} = 1. \tag{3.12}$$

This simplifies Einstein's field equations (3.7) considerably;

$$G_{11} = G_{22} = G_{33} = -\frac{1}{\mathcal{R}^2}, \quad G_{44} = -\frac{3}{\mathcal{R}^2}, \quad \kappa p = -\frac{1}{\mathcal{R}^2}, \quad \kappa\mu_0 = \frac{3}{\mathcal{R}^2}. \tag{3.13}$$

Thus, the EOS

$$\mu_0 + 3p = 0 \tag{3.14}$$

is the consequence of the field equations, describing a universe with non-accelerating expansion. Inserting this result into the conservation law (3.8) one finds consistence.

Cosmologists also try to present their models in non-comoving coordinates. Therefore, one has to set up a coordinate transformation to obtain a line element in non-comoving coordinates. Melia [5] made an attempt to derive such a line element, but he ended up with an equation containing non-comoving variables, but still containing the comoving time coordinate. A transformation matrix would be necessary to obtain from the Christoffel symbols (2.4) the corresponding Christoffel symbols in the non-comoving system. We doubt whether such a transformation exists.

Although a non-comoving observer can hardly be realized in Nature, a representation in a non-comoving system could give some insights into the structure of the model. It is possible to transform the tetrads (2.5) into a non-comoving system with a Lorentz transformation operating on the tetrad indices of (2.5) and maintaining the coordinate indices. With these 4-beine one can derive the Ricci-rotation coefficients for the non-comoving system. But it is more comfortable to use the inhomogeneous transformation law of the Ricci-rotation coefficients

$$A_{mn}{}^{s} = L^{m'n's}_{m\,n\,s'}{}' A_{m'n'}{}^{s'} + L^{s}_{s'} L^{s'}_{n|m}. \tag{3.15}$$

Here, the quantities L denote the Lorentz transformation. From now on we prime the kernels and indices for quantities of the comoving system. The Hubble law provides the relative velocity and the Lorentz factor of the reference systems:

$$v = Hr = \frac{1}{\mathcal{K}}\dot{\mathcal{K}} r = \frac{1}{\mathcal{R}}\dot{\mathcal{R}} r \, . \tag{3.16}$$

Applying (3.12), one has the simple relations

$$v = \frac{r}{\mathcal{R}}, \quad \alpha = \frac{1}{\sqrt{1-\frac{r^2}{\mathcal{R}^2}}} \, . \tag{3.17}$$

We note that in a flat infinite universe, the recession velocity v of the galaxies can exceed the velocity of light, and a Lorentz transformation has to be restricted to the region $r < r_H$, the region inside the cosmic horizon bounded by r_H . This is a hindrance in all open models. Applying (3.15) to the quantity U, we obtain an expression containing forces difficult to explain in the flat space scenario. We treat this problem in the following section. We emphasize that despite a change of notation and the introduction of new quantities, we still treat the Melia model.

4. SUBLUMINAL MODEL VS R_H=ct MODEL

In this section we compare Melia's R_h=ct model with our Subluminal Model. First, we continue with the problem of transforming the Ricci-rotation coefficients from a comoving reference system into a non-comoving reference system. We have already done this in our paper [2]:

$$U_m = \left\{-\alpha v \frac{1}{\mathcal{R}}, 0, 0, 0\right\} + \left\{i\alpha^3 v\, {}'U_4, 0, 0, -i\alpha^2\, {}'U_1\right\}, \quad {}'U_m = L_m^{m'} U_{m'} \, . \tag{4.1}$$

In the context of our positively curved Subluminal Model, we can explain this expression. Switching off the expansion (${}'U_m = 0$), one is left with

$$U_m = \left\{-\alpha v \frac{1}{\mathcal{R}}, 0, 0, 0\right\}. \tag{4.2}$$

This is a geometrical quantity and is just the force ${A_{41}}^4 = U_1$ one can derive from the dS metric, the seed metric of our Subluminal Model

$$ds^2 = \frac{1}{1-r^2/\mathcal{R}^2} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \left(1-r^2/\mathcal{R}^2\right) dt^2 \, . \tag{4.3}$$

The second brackets in (4.1) are the contributions of the expansion.

The dS metric can also be written in the form

$$ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \sin^2\eta d\vartheta^2 + \mathcal{R}^2 \sin^2\eta \sin^2\vartheta d\varphi^2 + \mathcal{R}^2 \cos^2\eta\, di\psi^2 \, , \tag{4.4}$$

representing the metric of a pseudo-hypersphere with constant radius $\mathcal{R}$. η is the polar angle and $a = 1/\alpha = \cos\eta = \sqrt{1-r^2/\mathcal{R}^2}$ the lapse function. Respecting these results, one is able to interpret the quantities introduced with (3.3). $\mathcal{R}$ is the time dependent radius of the pseudo-hypersphere envisaged in the non-comoving system, and $\mathcal{R}_0$ the very radius measured in the comoving system with expanding rods. In (3.13) we find expressions for pressure and mass density that have become familiar from other cosmological models.

From the relations $r = \mathcal{R}\sin\eta$, we find that the maximal value of r is $r_H = \mathcal{R}$, the equator of the pseudo-hypersphere relative to an observer who defines his position as pole. Thus, the cosmic horizon is a geometric property of the Subluminal Model.

Arriving at (3.13), the suspicion must arise that the R_h=ct model is based on a pseudo-hypersphere and is not globally flat but only locally flat. This is substantiated by the occurrence of geometrically interpretable quantities in a non-comoving system. Basically, the following considerations apply:

> If forces arise in a cosmological model, forces that originate from or act on masses, then the space has to be curved according to the general theory of relativity.

If this principle is disregarded by cosmology, it occupies a position that lies outside the large world of the rest of physics. Cosmology only then would tolerate violations of the rules of special relativity and allow acausalities.

We constructed the Subluminal Model based on the dS metric as seed metric, having the form parameter $k = 1$. Then we dropped the condition $\mathcal{R} = \text{const.}$ and we found a metric in comoving coordinates with $k = 0$, describing a cosmos expanding linearly in free fall. In contrast, examining Melia's model, one has to go in the transverse direction. We rewrite the metric (2.3) with

$$dx^1 = dr', \quad dx^2 = r'd\vartheta, \quad dx^3 = r'\sin\vartheta d\varphi$$

and retrieve Melia's original $k = 0$ metric

$$ds^2 = \mathcal{K}^2\left(dr'^2 + r'^2 d\vartheta^2 + r'^2\sin^2\vartheta d\varphi^2\right) - dt'^2 .$$

Then one has to transform the field strengths derived from this metric into a non-comoving system, review the methods and then find the dS metric as seed metric.

If we claim that the comoving coordinates $\{r', \vartheta, \varphi, t'\}$ in the line element above do not parametrize a flat space but are coordinates on a pseudo-hypersphere, both Melia's model and the Subluminal Model are geometrically identical in the end, but are presented in different ways. We contrast the two models in Table 1. This table gives an overview of the properties of the two models. On can find more details on this subject in our papers [8-11]. We emphasize that the differences between the two models are balanced, if one assumes that the form parameter of Melia's model can be interpreted for a model being positively curved and in free fall.

	R_h = ct Model	Subluminal Model
i.	globally flat	locally flat
ii.	recession velocity defined by the Hubble law	recession velocity geometrically defined
iii.	r is not bounded	r is bounded
iv.	superluminal recession velocities are possible	only subluminal velocities occur
v.	galactic islands are forming	galactic islands are not possible
vi.	the cosmic horizon is artificially implemented	the cosmic horizon is a geometric property of the model
vii.	the range of the Lorentz trans-formation has to be restricted	no restriction on the Lorentz transformation
viii.	the laws of special relativity are violated	the laws of special relativity are basic ingredients of the Subluminal Model
Ix.	the space is infinite	the space is finite
x.	the content of mass is infinite	the content of mass is finite

Table 1.

5. CONCLUSIONS

We analyzed Melia's R_h=ct model using a coordinate invariant notation. We showed that if we use this notation, the R_h=ct model and the Subluminal Model come closer and some contradictions can be removed. To avoid violation of the special theory of relativity, one has to reinterpret the R_h=ct model as a locally flat model expanding in free fall. Moreover, this reformulation reveals that conclusions which are drawn from some mathematical constructions have no physical meaning. The two models discussed are both exact solutions to Einstein's field equations, which complement each other and describe Nature better than the FRW models.

PEDAGOGICAL APPENDIX

Ricci-rotation coefficients vs Christoffel symbols

The Christoffel symbols are defined as

$$\Gamma^{l}_{ik} = \frac{1}{2} g^{ij}\left(g_{kj|i} + g_{ji|k} - g_{ik|j}\right), \tag{A}$$

the indices $i,k,...$ being coordinate indices. The Ricci-rotation coefficients have the form and symmetry property

$$A_{mn}{}^{s} = \overset{s}{e}_j \underset{[n|m]}{e^j} + g^{sr} g_{nt} \overset{t}{e}_j \underset{[m|r]}{e^j} - g^{sr} g_{mt} \overset{t}{e}_j \underset{[r|n]}{e^j}, \quad A_{m(ns)} = 0. \tag{B}$$

The two objects (A) and (B) are related by the inhomogeneous transformation

$$A_{nm}{}^{s} = \underset{n}{e^k} \underset{m}{e^i} \overset{s}{e}_j \Gamma^{j}_{ki} + \overset{s}{e}_j \underset{m|n}{e^j}. \tag{C}$$

$m,n,...$ are triad or tetrad indices. Evidently, the inhomogeneous term in Eq. (C) is highly coordinate dependent. Either the Christoffel symbols or the Ricci-rotation coefficients can represent physical quantities. We show with some examples that the Ricci-rotation coefficients are favored.

Case 1: Spherical coordinates

The line element in a 3-dimensional flat space in spherical coordinates reads as

$$ds^2 = dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 .$$

The metrical coefficients are

$$g_{rr} = 1, \quad g_{\vartheta\vartheta} = r^2, \quad g_{\varphi\varphi} = r^2 \sin^2\vartheta, \quad g^{rr} = 1, \quad g^{\vartheta\vartheta} = \frac{1}{r^2}, \quad g^{\varphi\varphi} = \frac{1}{r^2 \sin^2\vartheta}.$$

From these quantities we derive the Christoffel symbols

$$\begin{gathered} \Gamma^{\vartheta}_{r\vartheta} = \frac{1}{r}, \quad \Gamma^{r}_{\vartheta\vartheta} = -r, \quad \Gamma^{\varphi}_{r\varphi} = \frac{1}{r}, \quad \Gamma^{r}_{\varphi\varphi} = -r\sin^2\vartheta \\ \Gamma^{\varphi}_{\vartheta\varphi} = \cot\vartheta, \quad \Gamma^{\vartheta}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta \end{gathered} .$$

These are six different quantities without any discernible properties related to the geometry. In contrast we need to calculate only three quantities from the triads

$$\overset{1}{e}_1 = 1, \quad \overset{2}{e}_2 = r, \quad \overset{3}{e}_3 = r\sin\vartheta$$

to obtain the Ricci-rotation coefficients

$$
\begin{aligned}
A_{21}{}^{2} &= -A_{22}{}^{1} = -\overset{2}{e}_{2}\,\underset{2|1}{e^{2}} = \frac{1}{r} \\
A_{31}{}^{3} &= -A_{33}{}^{1} = -\overset{3}{e}_{3}\,\underset{3|1}{e^{3}} = \frac{1}{r} = \frac{1}{r\sin\vartheta}\sin\vartheta \\
A_{32}{}^{3} &= -A_{33}{}^{2} = -\overset{3}{e}_{3}\,\underset{3|2}{e^{3}} = \frac{1}{r}\cot\vartheta = \frac{1}{r\sin\vartheta}\cos\vartheta
\end{aligned}
\quad .
$$

Evidently, the Ricci-rotation coefficients describe the *curvatures* of the great circles and parallels of a 2-sphere. The associated *curvature radii* are $r, r\sin\vartheta$. This shows that the Ricci-rotation coefficients are related to geometrical quantities. Moreover, they simplify the calculations because the positions of lower and upper indices are arbitrary: $\Phi_m = \Phi^m$. Maintaining the original Minkowski notation, the timelike quantities do not change the sign by dragging indices.

The relation to the Christoffel symbols one obtains with Eq. (C). Labeling the spherical coordinates with $1 = r,\ 2 = \vartheta,\ 3 = \varphi$ one obtains

$$
\begin{aligned}
&A_{21}{}^{2} = \underset{2}{e^{2}}\,\underset{1}{e^{1}}\,\overset{2}{e}_{2}\,\Gamma^{2}_{21} = \frac{1}{r}, \quad A_{22}{}^{1} = \underset{2}{e^{2}}\,\underset{2}{e^{2}}\,\overset{1}{e}_{1}\,\Gamma^{1}_{22} = -\frac{1}{r} \\
&A_{31}{}^{3} = \underset{3}{e^{3}}\,\underset{1}{e^{1}}\,\overset{3}{e}_{3}\,\Gamma^{3}_{31} = \frac{1}{r} = \frac{1}{r\sin\vartheta}\sin\vartheta, \quad A_{33}{}^{1} = \underset{3}{e^{3}}\,\underset{3}{e^{3}}\,\overset{1}{e}_{1}\,\Gamma^{1}_{33} = -\frac{1}{r} = -\frac{1}{r\sin\vartheta}\sin\vartheta \\
&A_{32}{}^{3} = \underset{3}{e^{3}}\,\underset{2}{e^{2}}\,\overset{3}{e}_{3}\,\Gamma^{3}_{32} = \frac{1}{r}\cot\vartheta = \frac{1}{r\sin\vartheta}\cos\vartheta, \quad A_{33}{}^{2} = \underset{3}{e^{3}}\,\underset{3}{e^{3}}\,\overset{2}{e}_{2}\,\Gamma^{2}_{33} = -\frac{1}{r}\cot\vartheta = -\frac{1}{r\sin\vartheta}\cos\vartheta
\end{aligned}
\quad ,
$$

i.e., components which can be measured with rods in space.

Case 2: Schwarzschild exterior solution

The Schwarzschild exterior metric can be fully formulated with the curvatures of curves on a surface. The line element reads as

$$ds^2 = \rho_i\,\rho_k\,d\varepsilon^i d\varepsilon^k, \quad i = k\,,$$

with the following radii of curvature and angles

$$
\begin{gathered}
\rho_1 = \rho = \sqrt{\frac{2r^3}{M}}, \quad \rho_2 = r, \quad \rho_3 = r\sin\vartheta, \quad \rho_4 = \rho\cos\varepsilon \\
\varepsilon^1 = \varepsilon, \quad \varepsilon^2 = \vartheta, \quad \varepsilon^3 = \varphi, \quad \varepsilon^4 = i\psi
\end{gathered}
\quad .
$$

Written in full, the line element has the form

$$ds^2 = \rho^2 d\varepsilon^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta\, d\varphi^2 + \rho^2\cos^2\varepsilon\, di\psi^2\,.$$

Differentiating the curvature radius ρ of the Schwarzschild parabola, one obtains the relation between the coordinate r and the angle ε (with orientation cw), and lastly the curvatures quantities

$$A_{21}{}^{2} = \frac{1}{r}\cos\varepsilon, \quad A_{31}{}^{3} = \frac{1}{r\sin\vartheta}\sin\vartheta\cos\varepsilon, \quad A_{32}{}^{3} = \frac{1}{r\sin\vartheta}\cos\vartheta, \quad A_{41}{}^{4} = \frac{1}{\rho\cos\varepsilon}\sin\varepsilon\,.$$

The first three quantities describe the curvatures on Flamm's paraboloid, the last the force of gravity based on (open) pseudo circles with radii $\rho\cos\varepsilon$, foliating the Schwarzschild surface embedded in a 5-dimensional flat space. Using these quantities the Ricci can be split into subequations of type $\frac{\partial}{\partial r}\frac{1}{r}+\frac{1}{r^2}=0$.

It is left to the reader to translate the Christoffel symbols

$$\Gamma^{r}_{tt}=\frac{M}{r^3\left(r-2M\right)},\quad \Gamma^{r}_{rr}=-\frac{M}{r\left(r-2M\right)},\quad \Gamma^{t}_{tr}=\frac{M}{r\left(r-2M\right)}$$

$$\Gamma^{\vartheta}_{r\vartheta}=\frac{1}{r},\quad \Gamma^{\varphi}_{r\varphi}=\frac{1}{r},\quad \Gamma^{r}_{\vartheta\vartheta}=-r+2M$$

$$\Gamma^{r}_{\varphi\varphi}=-\left(r-2M\right)\sin^2\vartheta,\quad \Gamma^{\vartheta}_{\varphi\varphi}=-\sin\vartheta\cos\vartheta,\quad \Gamma^{\varphi}_{\vartheta\varphi}=\frac{\cos\vartheta}{\sin\vartheta}$$

into the above Ricci-rotation coefficients by applying Eq. (C).

Case 3: Kerr family

The Kerr geometry describing the exterior field of a rotating source is based on the seed metric

$$ds^2=\rho_S^2\,d\varepsilon^2+\rho_E^2\,d\theta_E^2+\rho_C^2\,d\varphi^2+\rho_\psi^2\,di\psi\,.$$

Here, ρ_S are the curvature radii of the radial lines of the Kerr surface. This surface is similar to Flamm's paraboloid, but elliptically squashed. ρ_E are the curvature radii of the ellipses, ρ_H the curvature radii of the hyperbolae – the orthogonal trajectories of the ellipses – , and $\rho_\psi=\rho_S\cos\varepsilon$ the curvature radii of (open) pseudo circles foliating the Kerr surface.

In some papers the Christoffel symbols are listed, filling pages. We are not prepared to perform calculations that neither seem useful nor mention the Christoffel symbols. Implementing the rotational effects of the Kerr metric, one gets a set of equations. In abridged notation this is

$$\begin{aligned}&\operatorname{div}\mathbf{F}=\mathbf{F}^2+2\mathbf{H}^2,\quad \operatorname{rot}\mathbf{F}=0\\ &\operatorname{div}\mathbf{H}=0,\qquad \operatorname{rot}\mathbf{H}=\mathbf{F}\times\mathbf{2H}\\ &\frac{\partial}{\partial t}\left(\mathbf{F}^2+2\mathbf{H}^2\right)=0,\quad \operatorname{div}\left(\mathbf{F}\times2\mathbf{H}\right)=0\end{aligned}\quad .$$

Here, $\mathbf{F}$ is the centrifugal force and $\mathbf{H}$ the Coriolis force. The structure of these equations is called gravitoelectromagnetism by some authors. We note that one is not able to set up these equations using the Christoffel symbols. Details can be found in [12][13]. One can derive similar structures for the NUT metric and the Kerr-Newman metric.

REFERENCES

[1] Melia F., *The lapse function in Friedmann-Lemaître-Robertson-Walker cosmologies.* Ann. Phys. **411**, 167997, 2019

https://doi.org/10.1016/j.aop.2019.167997

[2] Burghardt R., *Subluminal cosmology*. Journ. Mod. Phys. **8**, 583-601, 2017 https://doi.org/10.4236/jmp.2017.84039

[3] Melia F., Quin J., Zhang T. J., *Assessing cosmic acceleration with the Alcock- Paczyński effect in SDSS-IV quasar catalog.* MNRAS **499**, L36, 2020

https://doi.org/10.1093/mnrasl/slaa153

[4] Krasinski A., *Cosmological models and misunderstanding about them.* gr-qc/1110.1828

[5] Melia F., *The cosmic horizon*. MNRAS **382**, 1917, 2007

https://doi.org/10.1111/j.1365-2966.2007.12499.x

[6] Mitra A., *Deriving Friedmann Robertson Walker metric and Hubble's law from gravitational collapse formalism*. Results in Physics **2**, 45-49, 2012

http://dx.doi.org/10.1016/j.rinp.2012.04.002

[7] Mitra A., *Why the big bang model cannot describe the observed universe having pressure and radiation*. *Journ. Mod. Phys.* **2**, 1436-1442, 2011

http://doi.org/10.4236/jmp.2011.212177

[8] Burghardt R., *The curvature parameter in cosmological models.*

http://arg.or.at/Wpdf/Wcurv.pdf

[9] Burghardt R., *Generalization of the de Sitter cosmos.*

Journ. Mod. Phys. **9**, 685-700, 2018

https://doi.org/10.4236/jmp.2018.94047

[10] Burghardt R., *Local and global flatness in cosmology.*

Journ. Mod. Phys. **10**, 1439-1453, 2019

https://doi.org/10.4236/jmp.2019.1012096

[11] Burghardt R., *Melia's* R_h*=ct model is by no means flat.*

Journ. Mod. Phys. **11**, 703-711, 2020

https://doi.org/10.4236/jmp.2020.115045

[12] Burghardt R., *Spacetime Curvature.* (2020) http://arg.or.at/EMono.htm

[13] Burghardt R., *Raumkrümmung.* (2016) http://arg.or.at/Mono.htm