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COLLAPSING INTERIOR SCHWARZSCHILD SOLUTION, ABRIDGED VERSION

Rainer Burghardt*

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Abstract; We present an abridged version of our article 'Collapsing interior Schwarzschild solution' [1].

Since Oppenheimer and Snyder, inspired by an expanding cosmological model of Tolman in 1939, first proposed a model for a collapsing star, many authors have adopted this problem. Among the many suggestions are only a few exact solutions of Einstein's field equations. The reason is that the Einstein field equations are underdetermining and allthough the conservation laws have been consultedthere are not enough equations available to determine the metric coefficients and the physical quantities of the matter configuration.

Therefore we do not try to solve the Einstein field equations, but we construct the collapsing model with geometric methods. We begin with the static interior Schwarzschild solution and extend it to a collapsing model. We rely on the geometric interpretation of the interior and exterior Schwarzschild solutions as an embedding into a higher-dimensional flat space. The spatial part of the interior solution is a spherical cap which is joined to Flamm's paraboloid of the exterior solution in such a way that both surfaces have a common cutting tangent. The collapse takes place, if the spherical cap slides down the Schwarzschild parabola. The latter remains unchanged according to Birkhoff's theorem and essentially determines the course of the collapse.

A look at Fig. 1 shows that the model can be completely described by the inner surface Σ_i and the outer surface Σ_e of the whole Schwarzschild model. Thus, there is no surface Σ_c on which a line can be drawn and the elements of which can be attributed to a 'collapsing line element'. Nor is there any collapsing coordinate system. Since at any time of the collapse a snapshot of the collapsing object can be made, which is described by the interior Schwarzschild solution, one can access the coordinate system of Σ_i at any time.

* e-mail: arg@aon.at, home page: http://arg.or.at/

However, we do this only to determine the basic variables of the model. For all other investigations we use the tetrad calculus.

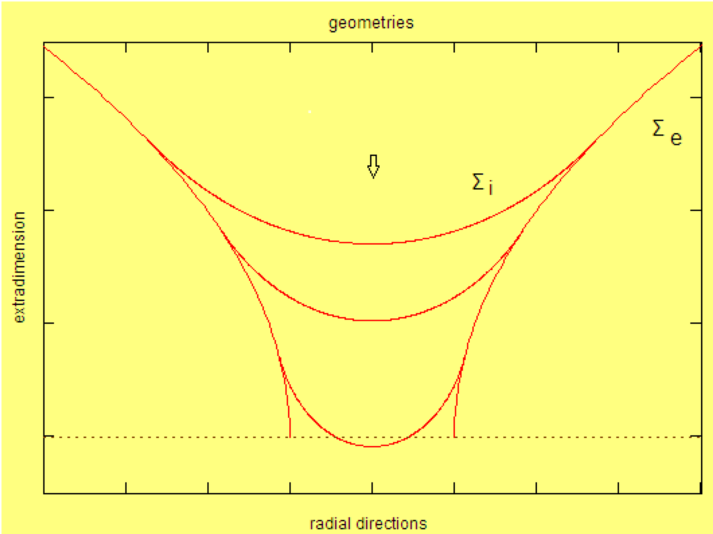

Fig. 1. Collapse of the interior

Both pressure and matter density of the collapsing model have the same analytical form as the static solution

$$\kappa p = -\frac{1}{\mathcal{R}_g^2}(1+2\mathfrak{p}), \quad \kappa\mu_0 = \frac{3}{\mathcal{R}_g^2}, \quad 1-\mathfrak{p} = \frac{3}{2}\frac{a_T^g}{a_T}, \quad p+\mu_0 = \frac{a_T^g}{a_T}\mu_0, \quad p = \left(\frac{a_T^g}{a_T}-1\right)\mu_0, \tag{1}$$

but vary with time. The stellar object can no longer be interpreted as an incompressible fluid sphere. $\mathcal{R}_g$ is the radius of the spherical cap. By matching the sphere to Flamm's paraboloid one obtains $\mathcal{R}_g = \rho_g/2$. ρ_g is the radius of curvature of the Schwarzschild parabola at the boundary of the interior and exterior geometries. Since it is easy to calculate it, one obtains $\mathcal{R}_g = \sqrt{r_g^3/2M}$, wherein r_g marks the position of the boundary surface in the embedding space and is time-dependent. It is advantageous to define the auxiliary variables

$$\mathfrak{p}_1^1 = \mathfrak{p}_2^2 = \mathfrak{p}_3^3 = 1, \quad \mathfrak{p}_4^4 = \mathfrak{p}, \quad \mathfrak{p} = -\frac{1}{2}\frac{a_R}{a_T}, \quad a_T = \frac{1}{2}\left(3a_R^g - a_R\right). \tag{2}$$

Therein is $a_R = \sqrt{1-r^2/\mathcal{R}_g^2}$ and a_R^g its value at the boundary surface. a_R and a_T are the metric coefficients of static interior solution

$$ds^2 = \alpha_R^2 dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - a_T^2 dt^2, \quad \alpha_R = 1/a_R. \tag{3}$$

A star described by this metric cannot be arbitrarily small. It has at $r_h = 9/4 \cdot M$ its minimum value, the *pressure horizon*. Having the extension r_h the pressure of the star would be infinite at its center. Further, if one let oscillate an object through the star, the object would reach the velocity of light in the center of the star if its extent has the minimum radius. Thus, there is also a *velocity horizon* which coincides with the pressure horizon. Therefore it can be assumed that a collapsing star can only shrink asymptotically to r_h. Thus, the model has an *inner horizon* which lies above the event horizon of the exterior solution. This will be an important property of our collapsing model.

We use two reference systems, the one (m) is in rest with respect to the exterior field and the other (m') is comoving with the collapse. We note a relation known from the static model and supplement it by an analogous for the comoving system:

$$\frac{1}{r} r_{|1} = \frac{a_R}{r}, \quad r_{|4} = 0, \quad \frac{1}{r'} r'_{|1'} = \frac{a_I}{r}, \quad r'_{|4'} = 0 . \tag{4}$$

The auxiliary variable r' with the range of values $[0,...,r'_g]$ is referred to in the literature as comoving radial coordinate. But we do not make use of this interpretation because we do not use or cannot use a coordinate system for the collapsing model. r'_g is the value of r' at the surface of the star. Furthermore, the following relations apply:

$$\begin{aligned} a_R^2 &= 1 - \frac{r^2}{\mathcal{R}_g^2}, \quad v_R = -\frac{r}{\mathcal{R}_g}, \quad \mathcal{R}_g = \sqrt{\frac{r_g^3}{2M}} \\ a_I^2 &= 1 - \frac{r'^2}{\mathcal{R}_0^2}, \quad v_I = -\frac{r'}{\mathcal{R}_0}, \quad \mathcal{R}_0 = \sqrt{\frac{r'^3_g}{2M}} \end{aligned} . \tag{5}$$

At the beginning of the collapse is $r_g = r'_g$ and $\mathcal{R}_g = \mathcal{R}_0$. We also demand that the two velocities v_R and v_I defined in (5) are composed to the collapse velocity according to Einstein's addition law of velocities

$$v_C = \frac{v_R - v_I}{1 - v_R v_I} . \tag{6}$$

With the relations above we have established the collapsing Schwarzschild model.

Between the two systems (m) and (m') the Lorentz transformation

$$L_{1'}^{1} = \alpha_C, \quad L_{1'}^{4} = i\alpha_C v_C, \quad L_{4'}^{1} = -i\alpha_C v_C, \quad L_{4'}^{4} = \alpha_C \tag{7}$$

mediates. With this and the inhomogeneous transformation law

$$'A_{m'n'}{}^{s'} = L_{m'n's}^{m\,n\,s'} A_{mn}{}^{s} + L_{s}^{s'} L_{n'|m'}^{s} . \tag{8}$$

one can transform the Ricci-rotation coefficients and thus the field strengths of the model from one system to the other. If we decompose the Ricci-rotation coefficients into

$$A_{mn}{}^{s} = B_{mn}{}^{s} + C_{mn}{}^{s} + U_{mn}{}^{s} \tag{9}$$

and further by use of the unit vectors

$$b_m = \{0,1,0,0\}, \quad c_m = \{0,0,1,0\}, \quad u_m = \{0,0,0,1\} \tag{10}$$

we lastly decompose into

$$B_{mn}{}^{s} = b_m B_n b^s - b_m b_n B^s, \quad C_{mn}{}^{s} = c_m C_n c^s - c_m c_n C^s, \quad U_{mn}{}^{s} = h_m{}^{s} U_n - h_{mn} U^s. \tag{11}$$

Therein

$$h_{mn} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \tag{12}$$

is the submetric of the tetrad metric $g_{mn} = diag(1,1,1,1)$. Thus, one has only to calculate the two lateral field quantities B and C and the acceleration U. With it one gains from the Ricci

$$R_{mn} = A_{mn}{}^{s}{}_{|s} - A_{n|m} - A_{rm}{}^{s} A_{sn}{}^{r} + A_{mn}{}^{s} A_s, \quad A_n = A_{rn}{}^{r} \tag{13}$$

the structure

$$\begin{aligned} R_{mn} &= -\left[U^s{}_{\underset{1}{||}s} + U^s U_s \right] h_{mn} \\ &\quad - \left[B_{n\underset{2}{||}m} + B_n B_m \right] - b_n b_m \left[B^s{}_{\underset{2}{||}s} + B^s B_s \right] \\ &\quad - \left[C_{n\underset{3}{||}m} + C_n C_m \right] - c_n c_m \left[C^s{}_{\underset{3}{||}s} + C^s C_s \right] \\ -\frac{1}{2} R &= \left[U^s{}_{\underset{1}{||}s} + U^s U_s \right] + \left[B^s{}_{\underset{2}{||}s} + B^s B_s \right] + \left[C^s{}_{\underset{3}{||}s} + C^s C_s \right] \end{aligned} \tag{14}$$

with the graded derivatives [2]

$$U_{n\underset{1}{||}m} = U_{n|m}, \quad B_{n\underset{2}{||}m} = B_{n|m} - U_{mn}{}^{s} B_s, \quad C_{n\underset{3}{||}m} = C_{n|m} - U_{mn}{}^{s} C_s - B_{mn}{}^{s} C_s. \tag{15}$$

The structure (14) is valid for all systems, for the static interior solution and the two systems of the collapsing model. We call the second term in (8) Lorentz term. It can be written as

$$'L_{m'n'}{}^{s'} = L_s^{s'} L^s_{n'|m'} = h_{m'}{}^{s'} \, 'L_{n'} - h_{m'n'} \, 'L^{s'}. \tag{16}$$

Evaluating it with (11), last equation, the inhomogeneous transformation law of the U-quantities is reduced to a vector equation

$$ {}'U_{m'} = U_{m'} + {}'L_{m'}, \quad {}'U_{m'} = \left\{ \mathfrak{p}\alpha_I v_I \frac{1}{\mathcal{R}_g}, 0, 0, -i\alpha_C v_C a_R \frac{1}{r} \right\}. \tag{17} $$

The primes in front of the kernel indicate that it is a quantity of the collapsing system, the primes on the indices that a quantity is measured in the comoving system. We have implemented the collapse by considering the field quantities as a function of time and by demanding that the primed reference system is connected to the collapse. After recourse to the static system we obtain

$$ B_{m'} = L^m_{m'} B_m = \left\{ \alpha_c \frac{a_R}{r}, 0, 0, -i\alpha_c v_c \frac{a_R}{r} \right\}, \quad C_{m'} = L^m_{m'} C_m = \left\{ \alpha_c \frac{a_R}{r}, \frac{1}{r}\cot\vartheta, 0, -i\alpha_c v_c \frac{a_R}{r} \right\} \tag{18} $$

and thus we have determined all field quantities of the comoving system.

It is essential for the collapse that the radius $\mathcal{R}_g$ of the spherical cap is time-dependent.Therefore the quantity

$$ \mathcal{F}_{1'} = 0, \quad \mathcal{F}_{4'} = \frac{1}{\mathcal{R}_g} \mathcal{R}_{g|4'} \tag{19} $$

occurs in our model. From the conservation law one gains

$$ \mu_{0|4'} = -(p + \mu_0)({}'U_{4'} + B_{4'} + C_{4'}) \overset{*}{=} -3(p + \mu_0){}'U_{4'}. $$

The fourth components of the quantities B, C, and 'U have equal values. This means that a volume element contracts equally in all three directions. From (1), second equation, one calculates $\mu_{0|4'} = -2\mu_0 \mathcal{F}_{4'}$ and finally one has the quantity

$$ \mathcal{F}_{4'} = (1 - \mathfrak{p}){}'U_{4'} = -i\alpha_C v_C (1 - \mathfrak{p}) \frac{a_R}{r} \tag{20} $$

which determines the course of the collapse. The conservation law is entirely treated with

$$ T^{m'n'}{}_{||n'} = 0, \quad p_{|1'} = -(p + \mu_0){}'U_{1'}, \quad \mu_{0|4'} \overset{*}{=} -3(p + \mu_0){}'U_{4'}. \tag{21} $$

If we determine the relations

$$B_{m'\underset{2}{\|}n'} = B_{m'|n'} - {}'U_{n'm'}{}^{s'}B_{s'}, \quad C_{m'\underset{3}{\|}n'} = C_{m'|n'} - {}'U_{n'm'}{}^{s'}C_{s'} - B_{n'm'}{}^{s'}C_{s'}$$

$$B_{m'\underset{2}{\|}n'} + B_{m'}B_{n'} = -\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \mathfrak{p} \end{pmatrix}\frac{1}{\mathcal{R}_g^2}, \quad B^{s'}{}_{\underset{2}{\|}s'} + B^{s'}B_{s'} = -(1+\mathfrak{p})\frac{1}{\mathcal{R}_g^2}$$

$$C_{m'\underset{3}{\|}n'} + C_{m'}C_{n'} = -\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & \mathfrak{p} \end{pmatrix}\frac{1}{\mathcal{R}_g^2}, \quad C^{s'}{}_{\underset{3}{\|}s'} + C^{s'}C_{s'} = -(2+\mathfrak{p})\frac{1}{\mathcal{R}_g^2}, \tag{22}$$

$${}'U^{s'}{}_{\underset{1}{\|}s'} + {}'U^{s'}{}'U_{s'} = -\frac{\mathfrak{p}}{\mathcal{R}_g^2}$$

we have completely eveluated the field equations if in addition we take into consideration the stress-energy tensor

$$T_{m'n'} = -pg_{m'n'} + (p+\mu_0){}'u_{m'}{}'u_{n'}. \tag{23}$$

To determine the field quantities for the non-comoving system, we use again the Lorentz transformation

$$B_m = L_m^{m'}B_{m'}, \quad C_m = L_m^{m'}C_{m'}, \quad U_m = L_m^{m'}{}'U_{m'} + L_m, \quad L_m = -L_m^{m'}{}'L_{m'}. \tag{24}$$

The lateral quantities take the static form. However, for the quantity U

$$U_m = -E_m - \alpha_R^2 v_R^2 \mathcal{F}_m, \quad \mathcal{F}_m = L_m^{m'}\mathcal{F}_{m'}, \quad E_m = \mathfrak{p}\alpha_R v_R \frac{1}{\mathcal{R}_g} \tag{25}$$

is valid. It can be seen that this expression only goes over into the static one, if one turns off the collapse, thus if one puts $\mathcal{R}_g = \text{const.}$, $(\mathcal{F}_m = 0)$. The U-equation is form-invariant

$$U^s{}_{\underset{1}{\|}s} + U^sU_s = -\frac{\mathfrak{p}}{\mathcal{R}_g^2}. \tag{26}$$

On the other hand we get for the B- and C-equations

$$B_{\underset{2}{m\|n}} + B_m B_n = -\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \mathfrak{p} \end{pmatrix} \frac{1}{\mathcal{R}_g^2} + \begin{pmatrix} -\hat{U}_1\mathcal{F}_1 & & & -\hat{U}_1\mathcal{F}_4 \\ & 0 & & \\ & & 0 & \\ -\mathcal{F}_4\hat{U}_1 & & & \hat{U}_1\mathcal{F}_1 \end{pmatrix}$$

$$C_{\underset{3}{m\|n}} + C_m C_n = -\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & \mathfrak{p} \end{pmatrix} \frac{1}{\mathcal{R}_g^2} + \begin{pmatrix} -\hat{U}_1\mathcal{F}_1 & & & -U_1\mathcal{F}_4 \\ & 0 & & \\ & & 0 & \\ -\mathcal{F}_4\hat{U}_1 & & & \hat{U}_1\mathcal{F}_1 \end{pmatrix}. \qquad (27)$$

$$B^s{}_{\underset{2}{\|s}} + B^s B_s = -(1+\mathfrak{p})\frac{1}{\mathcal{R}_g^2}, \quad C^s{}_{\underset{3}{\|s}} + C^s C_s = -(2+\mathfrak{p})\frac{1}{\mathcal{R}_g^2}$$

This allows us to represent the Einstein tensor completely. We only have to prepare the right side of the field equations. The stress-energy tensor in the non-comoving system

$$T_{mn} = -p g_{mn} + (p+\mu_0)'u_m{}'u_n, \quad 'u_m = \{-i\alpha_C v_C, 0, 0, \alpha_C\} \qquad (28)$$

we write component by component

$$\begin{aligned} &T_{11} = -p - \alpha_C^2 v_C^2 (p+\mu_0), \quad T_{22} = -p, \quad T_{33} = -p, \\ &T_{41} = -i\alpha_C^2 v_C (p+\mu_0), \quad T_{44} = \mu_0 + \alpha_C^2 v_C^2 (p+\mu_0) \end{aligned}. \qquad (29)$$

The question arises of whether the stress-energy tensor can be geometrized, ie whether the quantities of the right side of the field equations can be brought into connection with the very different field quantities of the left side. If we calculate the quantities in the second brackets of (27)

$$2\hat{U}_1\mathcal{F}_1 = \alpha_C^2 v_C^2 \kappa (p+\mu_0), \quad 2\hat{U}_1\mathcal{F}_4 = i\alpha_C^2 v_C \kappa (p+\mu_0) \qquad (30)$$

it is evident that the lateral subequations establish the necessary connection. If one also takes into consideration the U-equation, one has geometrized the stress-energy tensor.

Finally is to be investigated the final state of the star. We refer the collapse velocity $v_C = dx^1/dT$ to the surface of the star. At this location is $r = r_g, r'_g = r_0 = \text{const.}$. With the Lorentz relation $dT/dT' = \alpha_C$ we get

$$\frac{\alpha_R dr}{dT'} = \alpha_C v_C = \alpha_R \alpha_I (v_R - v_I), \quad dT' = \frac{1}{\alpha_I v_I} \frac{\sqrt{r}}{\sqrt{r_0} - \sqrt{r}} dr.$$

In it are α_I, v_I, and r_0 constants. Integration results in a function

$$f(r) = -\frac{1}{\alpha_l v_l}\left[r + 2\sqrt{r_0}\sqrt{r} + 2r_0 \ln\left(\sqrt{r_0} - \sqrt{r}\right)\right],$$

which is to be regarded in the range $[r_h, r_0]$. Since r is an outgoing coordinate the collapse, however, is directed inwards, we shift the origin of the coordinate system to the position of the surface, and that at the beginning of the collapse. We let run inwards the new radial coordinate $r \to r_0 - r$, $r_h \to r_0 - r_h$. Then we have

$$g(r) = r_0 - r + 2\sqrt{r_0 - r_h}\sqrt{r_0 - r} + 2(r_0 - r_h)\ln\left(\sqrt{r_0 - r_h} - \sqrt{r_0 - r}\right).$$

If we choose the constant of integration as $g(r_0) = 2(r_0 - r_h)\ln\sqrt{r_0 - r_h}$, then the proper time at the beginning of the collapse is $T' = 0$. Finally, one has in the the range under consideration

$$T'(r) = -\frac{1}{\alpha_l v_l}\left[g(r) - g(r_0)\right] . \tag{31}$$

The function is depicted in Fig. 2.

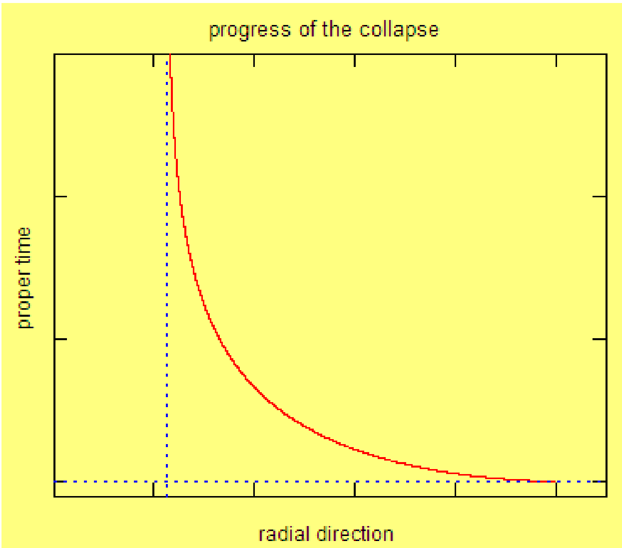

Fig. 2. Progress oft he collapse

From the figure one can gather how much time has passed, if the surface of the star has moved a certain distance $r_0 - r$. From $\lim_{r \to r_h} T'(r) = \infty$ is apparent that the star needs an infinitely long time to reach the minimum radius. Thus, the collapsing interior Schwarzschild solution has an *inner horizon*. It is identical to the above-mentioned pressure horizon and the velocity horizon. The star can never shrink to a point. The matter density, the pressure, and the curvature of space never are infinite. The inner horizon is above the event horizon of the exterior Schwarzschild solution.

Conclusions

The formation of a black hole in this model is not possible. It describes an ECO (eternally collapsing object), as it was predicted by Mitra [3] on the basis of astrophysical considerations. Since the exterior Schwarzschild solution has been proven and describes Nature well, one can assume that the interior solution can describe the interior of a star at least in a rough approximation. Although the two parameters, pressure and mass density are not sufficient to record the properties of a star, there is still hope that at least some basic properties of the model have general validity and that also more pretentious models do not exhibit unusual behaviors.

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