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REMARKS ON THE KRUSKAL METRIC

Rainer Burghardt[*]

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Abstract. We re-investigate the Kruskal metric by invoking two geometrical principles: we interpret the curvature literally as the curvature of a surface embedded in a higher dimensional flat space and we introduce the time as an imaginary variable. As a consequence of these formal presumptions the Kruskal metric cannot describe the interior region but matches a Lorentz transformation in the tangent spaces of the surface. This transformation is responsible for the acceleration of the observers in the Schwarzschild field. We also investigate the expanding 3-surface following the accelerated observers and identify the second fundamental forms of this surface as tidal forces.

* e-mail: arg@aon.at, home page: http://arg.or.at/

1. INTRODUCTION

In a previous paper [1] we have shown the different ways one could interpret curvature of space, and we have pointed out the consequences for the gravitation theory by applying these different interpretations of curvature.

We face the following different views on how to understand curvature:

(I) Spaces have curvature, if they are non-Euclidean. The space is not curved, but the geometry, the curvature is determined by the metric. A higher dimensional flat space for embedding the four-dimensional world does not exist. This point of view has the advantage that one can treat solutions of the Einstein field equations also for cases where surfaces do not exist.

(II) The geometry might be described by embedding. However, this is not necessary as the curvature can be expressed by the intrinsic properties of the 4-dimensional space.

(III) The geometry is explained by embedding surfaces into a higher dimensional flat space. The main advantage of this method is that one can utilize the tools of differential geometry like Gauss and Codazzi equations, which could give some insight into the geometrical structure of the model.

Whittaker in his textbook [2] has impressively formulated view (I)

"*Unhappily it has become important historically, for it has led to the creation of a terminology which is now so well established that we can never hope to change it, regrettable though it is, and which has been responsible for a great deal of popular misconception. The terminology in question is, that mathematicians apply the word 'curved' to any space whose geometry is not Euclidean. It is an unfortunate custom, because curvature, in the sense of bending, is a meaningless term except when the space is immersed in another space, whereas the property of being non-Euclidean is an intrinsic property which has nothing to do with immersion. However, nothing can be done but to utter a warning that what mathematicians understand by the term 'curvature' is not what the word connotes in ordinary speech: what the mathematician means is simply that the relations between mutual distances of the points are different from the relations which obtain in Euclidean geometry. Curvature (in the mathematical sense) has nothing to do with the shape of the space – whether it is bent or not – but is defined solely by the metric, that is to say, the way in which distance is defined. It is not the space that is curved, but the geometry of the space.*"

We have shown in some former papers that view (III) is very useful to describe the nature of gravitation. We have found new interior solutions by the use of the theory of surfaces and we have been able to explain the cosmological constant and the stress-energy tensor of gravitational models with the help of the second fundamental forms of surfaces.

There is another thing, where physicists are of different opinion, namely, how to implement the time into the metric. In flat space one can write the line element in two different ways

$$ds^2 = dx^\alpha dx^\alpha - dx^4 dx^4, \quad dx^4 = dt, \quad g_{44} = -1\ , \tag{1.1}$$

$$ds^2 = dx^\alpha dx^\alpha + dx^4 dx^4, \quad dx^4 = idt, \quad g_{44} = +1\ . \tag{1.2}$$

The first method we call t-notation, the second it-notation. Misner, Thorne an Wheeler note in their textbook [3] „*One sometime participant in special relativity will have to be put to the sword 'x^4=ict' . This imaginary coordinate was invented to make the geometry of spacetime look formally as little different as possible from the geometry of Euclidean space; to make a Lorentz transformation look on paper like a rotation; and to spare one the distinction that one otherwise is forced to make between quantities with upper indices (such as the components p^μ of the energy-momentum vector) and quantities with lower indices (such as the components p_μ of the energy-momentum 1-form). However, it is no kindness to be spared this latter distinction. Without it, one can not know whether a vector is meant or the very different geometric object that is a 1-form. Moreover, there is a significant difference between an angle on which everything depends periodically (a rotation) and a parameter the increase of which gives rise to ever-growing momentum differences (the 'velocity parameter' of a Lorentz transformation). If the imaginary time-coordinate hides from view the character of the geometric object dealt with and the nature of a parameter in the transformation, it also does something even more serious: it hides the completely different metric structure of ++++ geometry and -+++ geometry. In Euclidean geometry, when the distance between two points is zero, the two points must be the same point. In Lorentz-Minkowski geometry when the interval between two events is zero, one event may be on Earth and the other on a supernova in the galaxy M31, but their separation must be a null ray (peace of a light cone). The backward-pointing light cone at a given event contains all the events by which that event can be influenced. The forward-pointing light cone contents all events that it can influence. The multitude of double light cones taking off from all the events of spacetime forms an interlocking causal structure. This structure makes the machinery of the physical world function as it does. If in a region where spacetime is flat, one can hide this structure from view by writing* $(\Delta s)^2 = (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 + (\Delta x^4)^2$, *with* $x^4 = ict$, *no one has discovered a way to make an imaginary coordinate work in the general curved spacetime manifold. If '*$x^4 = ict$*' cannot be used there, it will not be used here."*

In contrast to the last argument we have used the it-notation in former papers throughout and we have given a satisfactory explanation for the geometric meaning of this notation. In the Gödel and de Sitter models the timelike line element can be written as

$$idt = \mathcal{R} di\psi\ .$$

This is the infinitesimal arc length of a pseudo circle on a pseudo sphere with radius $\mathcal{R}$. For the Schwarzschild model we have successfully used

$$idt = \rho di\psi\ ,$$

where ρ is the r-dependent curvature radius of the Schwarzschild parabola and $\rho di\psi$ the infinitesimal arc lengths of a family of pseudo circles. Moreover, one does not have to care for the sign of the time-like components of a tensor by dragging the indices.

Applying both view (III) and the it-notation to the Kruskal metric one gets a different interpretation of the physics of this model.

2. THE LORENTZ TRANSFORMATION

To begin with, we write down the Schwarzschild metric in Kruskal-Szekeres co-ordinates

$$ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} \left(du^2 - dv^2 \right) + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2 \ , \tag{2.1}$$

where the new variables are defined by

$$u = \sqrt{\frac{r}{2M} - 1} e^{\frac{r}{4M}} \mathrm{ch} \frac{t}{4M}, \quad v = \sqrt{\frac{r}{2M} - 1} e^{\frac{r}{4M}} \mathrm{sh} \frac{t}{4M} \tag{2.2}$$

for r > 2M and

$$u = \sqrt{1 - \frac{r}{2M}} e^{\frac{r}{4M}} \mathrm{sh} \frac{t}{4M}, \quad v = \sqrt{1 - \frac{r}{2M}} e^{\frac{r}{4M}} \mathrm{ch} \frac{t}{4M} \tag{2.3}$$

for r < 2M. Introducing the angle

$$\chi = \frac{t}{4M} \tag{2.4}$$

we go over to the it-notation by using trigonometric functions of an imaginary angle instead of hyperbolic functions. In addition, using view (III) we demand that the geometry is based on a surface as we have outlined in a previous paper [4]. As the surface has its boundary at r = 2M, one is not able to describe the inner region in the framework of surface theory. Thus, we have to search for another explanation for the sectors II an IV, described by (2.3). As we have already done in our paper [5], we extract the factor -1 from the roots of (2.3). In this way we force the range of the function to be r > 2M and we obtain the Kruskal metric in the form of

$$ds^2 = \gamma^2 \left(du^{1^2} + du^{4^2} \right) + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2 \ , \tag{2.5}$$

$$\gamma^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} \tag{2.6}$$

and the four Kruskal sectors

$$\mathrm{I} \quad \begin{matrix} u^1 = Y\cos i\chi \\ u^4 = Y\sin i\chi \end{matrix}, \quad \mathrm{II} \quad \begin{matrix} u^1 = \ \ Y\sin i\chi \\ u^4 = -Y\cos i\chi \end{matrix}, \quad \mathrm{III} \quad \begin{matrix} u^1 = -Y\cos i\chi \\ u^4 = -\ Y\sin i\chi \end{matrix}, \quad \mathrm{IV} \quad \begin{matrix} u^1 = -Y\sin i\chi \\ u^4 = \ \ Y\cos i\chi \end{matrix}. \tag{2.7}$$

Y is defined as

$$Y(r) = \frac{\sqrt{1-2M/r}}{\sqrt{2M/r}} e^{\frac{r}{4M}}, \quad Y(2M) = 0. \tag{2.8}$$

The pseudo circles $u^{1^2} + u^{4^2} = Y^2$ consist of four branches of hyperbolae of constant curvature and two null lines for r = 2M. Next, we relate a Lorentz transformation to (2.7). By differentiation of (2.7) and (2.8) for sector I we obtain

$$du^1 = \cos i\chi dY - Y \sin i\chi di\chi, \quad du^4 = \sin i\chi dY + Y \cos i\chi di\chi \tag{2.9}$$

and multiplying with γ we get the rotated vectors

$$dx^{1'} = \cos i\chi dx^1 - \sin i\chi dx^4, \quad dx^{4'} = \sin i\chi dx^1 + \cos i\chi dx^4, \tag{2.10}$$

from which we can read the components of the Lorentz transformation

$$L_1^{1'} = \cos i\chi, \quad L_4^{1'} = -\sin i\chi, \quad L_1^{4'} = \sin i\chi, \quad L_4^{4'} = \cos i\chi \tag{2.11}$$

with the velocity parameter

$$v_K = \mathrm{th}\chi\ . \tag{2.12}$$

In (2.10) we have used the Schwarzschild standard expressions for the radial line element and the local time interval

$$dx^1 = \frac{1}{\cos\varepsilon} dr, \quad dx^4 = \cos\varepsilon\, i dt, \quad \cos\varepsilon = \sqrt{1-2M/r}\ . \tag{2.13}$$

For sectors II and IV we obtain the velocity parameter for a tachyonic motion [6]

$$v_K = \mathrm{cth}\chi\ . \tag{2.14}$$

Since Lorentz transformations can be drawn as hyperbolae in pseudo-real representation the Kruskal diagram can be interpreted as a diagram for bradyonic and tachyonic Lorentz transformations.

3. THE TIDAL FORCES

In our previous paper on the Kruskal metric [5] we have discussed the above problem in more detail and we also have investigated the field equations with the help of the [3+1] decomposition of spacetime. In this paper we analyze the forces acting on a Kruskal-accelerated observer. We have outlined that the transition to the accelerated system does not alter the underlying geometry anyway. The Lorentz transformation accompanying the Kruskal co-ordinate transformation is a pseudo rotation in the tangent spaces of a 4-surface. Now we envisage the expanding 3-surface following an observer on his way, being accelerated outwards. We derive the forces acting on him with the method of the second fundamental forms of the 3-surface. We proceed in a way similar to that in which we have treated freely falling observers in the Schwarzschild field [7].

Transforming the covariant derivatives from Schwarzschild static system to the Kruskal reference system we have

$$\Phi_{m'||n'} = L^{m\,n}_{m'n'}\Phi_{m||n} = \Phi_{m'|n'} - A_{n'm'}{}^{s'}\Phi_{s'} - L_{n'm'}{}^{s'}\Phi_{s'}, \quad L_{n'm'}{}^{s'} = L^{s'}_{s} L^{s}_{m'|n'} \;. \tag{3.1}$$

Calculating the Lorentz term L with

$$\partial_{1'} = \cos i\chi\,\partial_1 - \sin i\chi\,\partial_4, \quad \partial_{4'} = \sin i\chi\,\partial_1 + \cos i\chi\,\partial_4 \tag{3.2}$$

we obtain

$$\begin{gathered} L_{m'n'}{}^{s'} = h_{m'n'}K^{s'} - h_{m'}{}^{s'}K_{n'}, \quad h_{m'n'} = {}'m_{m'}{}'m_{n'} + {}'u_{m'}{}'u_{n'} = m_{m'}m_{n'} + u_{m'}u_{n'} \\ K_{n'} = \frac{1}{4Ma}m_{n'} = \frac{1}{4Ma}\{\cos i\chi, 0, 0, \sin i\chi\} \end{gathered} \;, \tag{3.3}$$

wherein the radial and time-like unit vectors of the Kruskal system are

$${}'m_{n'} = \{1,0,0,0\}, \quad {}'u_{m'} = \{0,0,0,1\} \tag{3.4}$$

and the unit vectors of the static system measured by the Kruskal observers

$$m_{n'} = \{\cos i\chi, 0, 0, \sin i\chi\} \quad u_{m'} = \{-\sin i\chi, 0, 0, \cos i\chi\} \;. \tag{3.5}$$

The decomposition of the Ricci rotation-coefficients leads us to a set of field strengths

$$B_{m'} = \left\{ \frac{a}{r} \cos i\chi, 0, 0, \frac{a}{r} \sin i\chi \right\}$$

$$C_{m'} = \left\{ \frac{a}{r} \cos i\chi, \frac{1}{r} \cot \vartheta, 0, \frac{a}{r} \sin i\chi \right\} , \tag{3.6}$$

$$E_{m'} = \left\{ \frac{1}{\rho} \frac{v}{a} \cos i\chi, 0, 0, \frac{1}{\rho} \frac{v}{a} \sin i\chi \right\}$$

describing the static system, but with the values measured in the accelerated system. The Lorentz term is incorporated in the graded derivative

$$\Phi_{m' \underset{1}{\|} n'} = \Phi_{m'|n'} - L_{n'm'}{}^{s'} \Phi_{s'} , \quad L_{n'm'}{}^{s'} = L^{s'}_{s} L^{s}_{m'|n'} . \tag{3.7}$$

The definitions for the graded covariant derivatives and more on this technique one can find in our papers cited below. The field equations for these quantities are

$$R_{n'm'} = -\left[B_{m' \underset{2}{\|} n'} + B_{m'} B_{n'} \right] - b_{m'} b_{n'} \left[B^{r'}{}_{\underset{2}{\|} r'} + B^{r'} B_{r'} \right]$$

$$- \left[C_{m' \underset{3}{\|} n'} + C_{m'} C_{n'} \right] - c_{m'} c_{n'} \left[C^{r'}{}_{\underset{3}{\|} r'} + C^{r'} C_{r'} \right]$$

$$+ \left[E_{m' \underset{4}{\|} n'} - E_{m'} E_{n'} \right] + u_{m'} u_{n'} \left[E^{r'}{}_{\underset{4}{\|} r'} - E^{r'} E_{r'} \right] = 0 . \tag{3.8}$$

$$B_{[m' \underset{2}{\|} n']} = 0, \quad C_{[m' \underset{3}{\|} n']} = 0, \quad E_{[m' \underset{4}{\|} n']} = 0$$

To obtain the equations for the forces acting on the accelerated observers one has to start with the Ricci

$$R_{m'n'} = {}'A_{m'n'}{}^{s'}{}_{|s'} - {}'A_{n'|m'} - {}'A_{r'm'}{}^{s'} {}'A_{s'n'}{}^{r'} + {}'A_{m'n'}{}^{s'} {}'A_{s'}$$

$${}'A_{n'm'}{}^{s'} = L^{n\,m\,s'}_{n'm's} A_{nm}{}^{s} + L_{n'm'}{}^{s'} . \tag{3.9}$$

Re-arranging the time-like part of the connexion coefficients

$$E_{m'n'}{}^{s'} = h_{m'n'} E^{s'} - h_{m'}{}^{s'} E_{n'} , \tag{3.10}$$

we obtain by contraction of this quantity and with (3.3) the effective Kruskal force

$$K^{e}_{m'} = K_{m'} + E_{m'} . \tag{3.11}$$

Separating all time-like parts from the connexion coefficients we get the second fundamental forms of the expanding 3-surface

$$Q_{1'1'} = A_{1'4'}{}^{1'} + L_{1'4'}{}^{1'} = -K^{e}_{4'}, \quad Q_{2'2'} = B_{4'} = \frac{a}{r}\sin i\chi, \quad Q_{3'3'} = C_{4'} = \frac{a}{r}\sin i\chi \ . \tag{3.12}$$

The first expression contains a dynamical contribution due to the Lorentz transformation. The Q’s are the tidal forces acting on the accelerated system. Dropping the primes of the indices we write for the connexion coefficients

$$A_{mn}{}^{s} = {}^{*}A_{mn}{}^{s} + Q_{m}{}^{s}\,{}'u_{n} - Q_{mn}\,{}'u^{s} \ . \tag{3.13}$$

We obtain the [3+1] decomposition of the Ricci with respect to the accelerated system

$$\begin{aligned} R_{mn} = & -\left[{}^{*}B_{n\underset{2}{\wedge}\underline{m}} + {}^{*}B_{n}{}^{*}B_{m} \right] - b_{m}b_{n}\left[{}^{*}B^{s}{}_{\underset{2}{\wedge}s} + {}^{*}B^{s}{}^{*}B_{s} \right] \\ & -\left[{}^{*}C_{n\underset{3}{\wedge}\underline{m}} + {}^{*}C_{n}{}^{*}C_{m} \right] - c_{m}c_{n}\left[{}^{*}C^{s}{}_{\underset{3}{\wedge}s} + {}^{*}C^{s}{}^{*}C_{s} \right] \\ & +\left[{}^{*}K^{e}_{n\underset{4}{\wedge}\underline{m}} + {}^{*}K^{e}_{n}{}^{*}K^{e}_{m} \right] + {}'u_{m}{}'u_{n}\left[{}^{*}K^{s}_{e\underset{4}{\wedge}s} + {}^{*}K^{s}_{e}{}^{*}K^{e}_{s} \right] \\ & -\left[Q_{mn|s}\,{}'u^{s} + Q_{mn}Q_{r}{}^{r} \right] \\ & -{}'u_{m}\left[{}^{*}B_{n|s}\,{}'u^{s} + {}^{*}B_{sn}{}^{r}Q_{r}{}^{s} + {}^{*}K^{e}_{n}Q_{22} \right] - {}'u_{m}\left[{}^{*}C_{n|s}\,{}'u^{s} + {}^{*}C_{sn}{}^{r}Q_{r}{}^{s} + {}^{*}K^{e}_{n}Q_{33} \right] \\ & -{}'u_{n}\left[Q_{s}{}^{s}{}_{\underset{4}{\wedge}\underline{m}} - Q_{m}{}^{s}{}_{\underset{4}{\wedge}s} \right] \\ & -{}'u_{m}{}'u_{n}\left[Q_{s}{}^{s}{}_{|r}\,{}'u^{r} + Q^{rs}Q_{rs} \right] = 0 \end{aligned} \ . \tag{3.14}$$

All quantities with an asterisk are 3-dimensional quantities and underlined indices are space-like. The graded ‘hat-derivative’ is constructed in analogy to the ‘double-stroke derivative’ by the use of the space-like connexion coefficients *A. Evidently, the contracted Codazzi equation

$$Q_{[s}{}^{s}{}_{\underset{4}{\wedge}\underline{m}]} = 0 \tag{3.15}$$

decouples from the field equations.

We have shown that the Kruskal metric describes the same surface as the Schwarzschild metric in standard Schwarzschild co-ordinates. However, the Kruskal metric gives us the hint to investigate a reference system naturally connected with the Kruskal co-ordinate system. Thus, an additional structure is implemented on the surface giving raise to new forces, the acceleration and the tidal forces. The price to pay is that we have to dismiss all speculations on black holes and the possibility of an observer passing the event horizon.

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