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COORDINATE SYSTEMS, REFERENCE SYSTEMS, BLACK HOLES, ECOS, AND COSMOLOGICAL PROBLEMS

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Abstract: In this paper we intend to show that the search for new coordinates to penetrate the event horizon of the Schwarzschild model, is hopeless. We also show that the formation of Black Holes is impossible in the framework of the complete Schwarzschild model, consisting of the exterior and interior solutions. In addition, we make some critical remarks about coordinate systems in cosmology.

Contents

1. Introduction 2
2. The Schwarzschild problem 2
3. Gravitatioal collapse 10
4. Cosmologic problems 12
5. Conclusions 16
6. Mathematical appendix 16
7. References 20

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1. INTRODUCTION

A fundamental principle of Einstein's theory of relativity is the independence of physical statements from a particular coordinate system. In principle, a coordinate system can be chosen arbitrarily, but it must be adapted to the physical situation in such a way that the calculation of a model is as simple as possible. Nevertheless, different coordinate systems are considered for the Schwarzschild model. This is done in the hope of gaining new insights, mainly to find a way to penetrate the event horizon. Also in cosmology, two different coordinate systems are used; one that comoves with the expansion of the cosmos and one that does not.

We will show that new coordinate systems are obsolete because they do not change the physical facts of the model and that a Lorentz transformation can be associated with them that describes a new state of motion of the observers. To find out what is left of the physics after a coordinate transformation, we use tetrads, i.e., local orthogonal 4-bein vectors representing rods and clocks. These are used to measure physical and geometric quantities. Tetrads and quantities are parallel-transported in the curved space with the help of Ricci-rotation coefficients. The use of the original Minkowski notation $x^4 = i(c)t$ is mandatory. We also reject the use of Christoffel symbols. They are highly coordinate-dependent and lead to mathematical artifacts. In the following sections, we focus mainly on the Schwarzschild model, but we also make some comments on the cosmological models.

2. THE SCHWARZSCHILD PROBLEM

We recall that the solution, proposed by Schwarzschild for the field of a non-rotating mass is not given in its original form today, but in Hilbert's notation. The line elements of both representations differ in the choice of the coordinate system. Both metrics can be written in the form

$$ds^2 = \frac{1}{1-\frac{2M}{r}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \left(1-\frac{2M}{r}\right) dt^2 . \tag{2.1}$$

However, in the original form of the Schwarzschild metric, r has the following meaning

$$r^3 = r_*^3 + 2M^3 , \tag{2.2}$$

whereby the radial coordinate is defined by $r_* = \sqrt{x^{\alpha'} x^{\alpha'}}$, wherein $x^{\alpha'}$ are Cartesian coordinates in the embedding space. $r_* = 0$ determines the origin of the coordinate system. At this point one has $r = 2M$, and the radial arc element is singular. Thus, the singularity occurs at the origin of the original Schwarzschild coordinate system and (2.1) is interpreted as the metric of a mass point. We are looking at the lateral arc element related to the polar angle ϑ. Substituting (2.2) into (2.1), we get $\sqrt[3]{r_*^3 + 2M^3}\, d\vartheta$. We note that for this arc element, the expression $2M d\vartheta$ is valid at the location $r_* = 0$. Therefore, a radius $2M$ must be assigned to a *mass point*. This is a discrepancy that can be corrected using the

Hilbert form of the metric. In this case, $r=0$ is the origin of the coordinate system. A singularity occurs at $r=2M$ and this location is called the event horizon.

We recall that **Droste** [1] also found a metric describing the Schwarzschild field and **Gullstrand** [2] presented a more general metric for a spherically symmetric system, which can be reduced to

$$\begin{gathered} ds^2 = dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) + 2iv\,dr\,i\,dt - a^2dt^2 \\ v = -\sqrt{2M/r}, \quad a = 1/\alpha = \sqrt{1-2M/r} \end{gathered} . \tag{2.3}$$

Painlevé [3] found this metric independently of Gullstrand. Obviously, the coordinate system used is oblique-angled. The time coordinate is not orthogonal to the space and, thus, is unphysical. Nevertheless, orthogonal tetrads can be read from (2.3) in two ways (Mathematical Appendix, formulae A1 and A2).

Using the Ricci-rotation coefficients, one obtains from (A1) the same field quantities that one would obtain from the Hilbert form of the metric (2.1). From (A2) one gets the field quantities for a freely falling system. We will revisit this problem later.

The calculations would be easier if the time coordinate was changed with

$$dt = dt' - \alpha^2 v dr .$$

With this transformation, one retrieves the metric in the form (2.1). It turns out that the Gullstrand-Painlevé coordinates are obsolete. More on this topic can be found in our booklet [4].

Of interest is a coordinate transformation by **Lemaître** [5]

$$r = \sqrt[3]{2M}\left[\frac{3}{2}(r'-t')\right]^{\frac{2}{3}}, \quad t'' = t + 2\sqrt{2Mr} + 2M\ln\frac{1-\sqrt{\frac{2M}{r}}}{1+\sqrt{\frac{2M}{r}}} . \tag{2.4}$$

When introducing a new quantity[1]

$$\mathcal{R}(r',t') = \frac{3}{2}(r'-t'), \quad r^3 = 2M\mathcal{R}^2, \tag{2.5}$$

with the familiar definition of the velocity of free fall in the Schwarzschild field $v=-\sqrt{2M/r}$ one obtains the Lemaître metric

$$ds^2 = v^2(r',t')\left[dr'^2 + \mathcal{R}^2 d\vartheta^2 + \mathcal{R}^2\sin^2\vartheta d\varphi^2\right] - dt'^2 . \tag{2.6}$$

The spatial coefficients of this metric have a time-dependent factor. Thus, the coordinate system shrinks in the direction of the field producing mass. The lapse function of the Lemaître metric is 1. Thus, the coordinate time t' is also the proper time and a universal time for all observers. Obviously, the velocity of a free-falling observer at the event horizon is $v=1$, the velocity of light in natural coordinates.

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[1] The new quantity $\mathcal{R}$ can be interpreted geometrically. If one extends the curvature vector of the Schwarzschild parabola to its guideline, the distance $\mathcal{R}$ between the guideline and the Schwarzschild parabola is cut out. For $r=2M$ also is $\mathcal{R}=2M$ and $\mathcal{R}$ is the radius of the circle at the waist of the Schwarzschild parabola.

In [4], we explicitly gave the transformation coefficients for a transformation from standard Schwarzschild coordinates to Lemaître coordinates. Further, we can read the tetrads from the Lemaître metric and calculate the field quantities. It is possible to associate a Lorentz transformation $\{\alpha, -i\alpha v, i\alpha v, \alpha\}$ with the Lemaître coordinate transformation. With this and the help of the inhomogeneous transformation law of the Ricci-rotation coefficients, we transform the Schwarzschild field quantities into the Lemaître field quantities.

Since the lapse function of the metric (2.6) is constant, no force of gravity can be derived from it. This is consistent with Einstein's elevator principle: freely falling observers cannot observe gravity. It should be noted that tidal forces [6] act on observers in all directions of space. Using the tetrad method, it is easy to show how the components of the 4-vectors derived from the Ricci-rotation coefficients are relocated under the influence of the Lorentz transformation

$$U_m = \{U_1, 0, 0, 0\} \rightarrow U_m = \{0, 0, 0, U_4\}. \tag{2.7}$$

U_1 refers to the force of gravity, and U_4 refers to the tidal forces. The left side of (2.7) refers to the standard Schwarzschild representation and the right side refers to the Lemaître representation. We addressed the problem of free fall in [7] and extended the problem to free fall from an arbitrary position in [8].

Although the Lemaître coordinate system is obsolete for describing Nature, it has some interesting features. Rewriting the standard Schwarzschild metric by introducing the angle of ascent ε with orientation cw [4]:

$$r = \mathcal{R}\sin\varepsilon, \quad v = \sin\varepsilon, \quad a = \cos\varepsilon, \tag{2.8}$$

one can write (2.1) in the canonical form

$$ds^2 = \frac{1}{1 - \frac{r^2}{\mathcal{R}^2}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - a^2 dt^2. \tag{2.9}$$

In cosmology, one classifies the metric with the curvature parameter k, writing for the radial coefficient

$$\frac{1}{1 - k\frac{r^2}{\mathcal{R}^2}}.$$

A metric with $k = \{1, 0, -1\}$ should describe a positively curved, flat, or negatively curved space. When applied to the Schwarzschild metric, we find that $k = 1$ for the static case but $k = 0$ for the case of free fall. Here, $k = 0$ does not mean that the space is flat. The geometrized basis of the model is still the Schwarzschild parabola. The Lorentz transformation to the Lemaître form of the metric does not change the curvature of space; it rotates the tetrads *on* the space. Therefore, we prefer to call k the *form parameter* of the metric.

The static coordinate system (2.1) is singular at $r = 2M$. This singularity can be removed by an appropriate coordinate transformation. But some remarkable properties of the model remain at the event horizon and cannot be changed by any coordinate transformation: for instance the velocity of an infalling observer reaches the velocity of light, and the force of gravity becomes infinite.

Next, we consider the **Einstein-Rosen** coordinates [9] introduced by the authors In [10]. Starting with the Schwarzschild parabola $R^2 = 8M(r-2M)$, we substitute $r = \frac{R^2+16M^2}{8M}$ into (2.1). Thus, for the Schwarzschild metric we get

$$ds^2 = \frac{R^2+16M^2}{16M^2} dR^2 + \left(\frac{R^2+16M^2}{8M}\right)^2 \left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right) - \frac{R^2}{R^2+16M^2} dt^2 . \qquad (2.10)$$

Here, R is the coordinate of the extra dimension in the flat embedding space and $R = 0$ corresponds to $r = 2M$ and denotes the event horizon. Note that the Einstein-Rosen coordinates cannot describe the interior region $r < 2M$ of the Schwarzschild model and all attempts to cross the event horizon fail.

Einstein and Rosen also considered the negative branch of the Schwarzschild parabola. The location $R = 0$ is called Einstein-Rosen Bridge, which connects the two regions $\pm R$. This gave rise to the speculation that the bridge could connect distant regions of the universe or could be an entrance to parallel universes. Wormholes occupy a large place in scientific and Sci-Fi literature. We recall that the force of gravity blows up at the bridge. Also, it takes an infinitely long proper time for any object to reach the bridge. In the Mathematical Appendix we considered the free fall from an arbitrary position and calculated the proper fall time diverging at $R = 0$ in Einstein-Rosen coordinates.

Other coordinate systems that cannot describe the interior of the Schwarzschild metric are **isotropic coordinate systems**. We studied them in [4,11]. Using the regular nonlinear transformation

$$r = \left(1 + \frac{M}{2\bar{r}}\right)^2 \bar{r} , \qquad (2.11)$$

where r is the radial Schwarzschild standard coordinate, one obtains the line element of the form

$$ds^2 = \left(1+\frac{M}{2\bar{r}}\right)^4 \left(d\bar{r}^2 + \bar{r}^2 d\vartheta^2 + \bar{r}^2 \sin^2\vartheta\, d\varphi^2\right) - \left(\frac{1-\frac{M}{2\bar{r}}}{1+\frac{M}{2\bar{r}}}\right)^2 dt^2 \qquad (2.12)$$

in isotropic coordinates, which are valid for all $\bar{r}$ of $0 < \bar{r} \le \infty$. The metric is regular everywhere throughout this range. The function $r(\bar{r})$ has a minimum at the event horizon $\bar{r}_H = M/2$ corresponding to $r = 2M$. The region $0 \le r < 2M$ is excluded, from the outset, by isotropic coordinates. More informatiion on this subject, can be found in the Mathematical Appendix. The evaluation of the field quantities, leads to the well-known standard Schwarzschild expressions, formulated in isotropic coordinates [11].

Eddington [12] and, later on, **Finkelstein** [13] introduced a new coordinate system which is free of singularities. The transformations with Eddington coordinates $\{r,t\}$ and standard Schwarzschild coordinates $\{r',t'\}$,

$$dt = dt' + \frac{2M}{r-2M} dr', \quad dt = dt' - \frac{2M}{r-2M} dr', \quad r = r' \qquad (2.13)$$

lead to the singularity-free form of the metric

$$ds^2 = dr^2 - dt^2 + v^2\left(dr + dt\right)^2 + d\Omega^2, \quad d\Omega^2 = r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2 . \tag{2.14}$$

Here, $v = \sqrt{2M/r}$ is the velocity of a freely falling observer. It is not surprising that the transformation (2.11) produces a metric free of singularities, because the transformation itself has a singularity at $r = 2M$. This is seldom mentioned in the literature. The integration of (2.13) gives

$$t = t' \pm 2M \ln\left(r - 2M\right). \tag{2.15}$$

The expression diverges at $r = 2M$.

The metric (2.14) can also be written in the form

$$ds^2 = \left(1 + v^2\right) dr^2 + 2v^2 dr\, dt - a^2 dt^2 . \tag{2.16}$$

It is not symmetric under time reversal, even though, Einstein's field equations have this symmetry. It also has a cross term, which indicates an oblique-angled coordinate system. We can write the metric inthe form

$$ds^2 = \alpha^2 dr^2 + \left(iadt - i\alpha v^2 dr\right)^2 . \tag{2.17}$$

In the Appendix, we have noted the 4-bein associated with this metric and also the transformation matrix to the Schwarzschild system, ready for the calculation of the field quantities and field equations. We, again, get the familiar structure of the Schwarzschild model.

Kruskal [14] and **Szekeres** [15] published a coordinate system covering four sectors of space. Many researchers thought that these coordinates could be used to inspect the interior region of the Schwarzschild model. The Kruskal metric

$$ds^2 = \gamma^2\left(du^{1^2} + du^{4^2}\right) + r^2 d\vartheta^2 + r^2 \sin^2\vartheta\, d\varphi^2, \quad \gamma^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} \tag{2.18}$$

is illustrated by the well-known Kruskal diagram. Two of the four sectors describe the conventional Schwarzschild theory; the other two are interpreted as a Black Hole or a white hole. However, we have shown in papers [16,17] that nothing new is gained by introducing the Kruskal coordinates. The coordinate transformation is associated with a Lorentz transformation with $\left\{\cos i\chi, -\sin i\chi, \sin i\chi, \cos i\chi\right\}$, and from this, we extract the velocities $v = \mathrm{th}\chi$ and $v = \mathrm{cth}\chi$ for sectors I, III and II, IV respectively. The latter is tachyonic. The expressions for the outgoing observer are obtained from the incoming observer, by time reversal, via the Lorentz angle $\chi \to -\chi$. Here, $\chi = t/4M$ is the scaled coordinate time (rapidity). In the Appendix (Fig. A3), we have shown the time course of the two velocities concerning the tachyonic and bradyonic sectors.

Since the Kruskal velocities are not constant, accelerations occur. The associated forces have to be subtracted from or added to the gravitational force depending on looking at an outgoing or incoming observer. The situation was illustrated in [4] and the field quantities were discussed in detail. It was shown that falling below the event horizon is not possible.

Markley [18] used the transformation to **isothermal coordinates**

$$\bar{r} = r + 2M\ln\left(\frac{r(\bar{r}) - 2M}{a - 2M}\right),$$

where a is a constant. He obtained the metric

$$ds^2 = \left(1 - \frac{2M}{r(\bar{r})}\right)\left(d\bar{r}^2 - dt^2\right) + r^2(\bar{r})d\vartheta^2 + r^2(\bar{r})\sin^2\vartheta d\varphi^2 . \tag{2.19}$$

The new radial coordinate $\bar{r}$ passes through the interval $[-\infty,+\infty]$ where the standard Schwarzschild coordinate passes through the range $\left[2M,\infty\right]$. The relation between the two coordinates can be seen in Fig. A4 of the Appendix. By evaluating the tetrads from (2.19), one obtains the well-known expressions for the Schwarzschild field equations.

Misner, Thorne, and Wheeler (MTW) calculated the free fall of an object falling from an arbitrary position in their textbook [19]. Unfortunately, the erroneous results of these calculations greatly influenced the development of physics. Apparently, it was the authority of these three researchers that prevented people from re-examining the problem. The authors note that a freely falling object asymptotically approaches the event horizon as seen from an observer at infinity, while a comoving observer crosses the horizon after a relatively short time. This is worse than with Schrödinger's cat. For one observer it lives forever, but for the other, it dies quickly.

Using Ockham's razor, at least four pages of MTW are reduced to five lines giving the same results. But the short representation shows the error: the proper length measured by an observer falling from a finite position, has been combined by MTW with the proper time of an observer coming from infinity. So they got[2]

$$ds^2 = dx'^2 - dT''^2 .$$

We addressed this problem in a talk in Berlin a few years ago. An illustration of this can be found in the English translation of the paper in [20]. There, the MTW method is compared with the strictly relativistic one.

The formulae, resulting from the MTW approach, contradict the basic formulae of the theory of relativity. As an example, we give the relation for the velocity of free fall as

$$v' = \sqrt{\frac{v^2 - v_0^2}{1 - v_0^2}} .$$

Here, v is the velocity of an observer coming from infinity, v_0 is the velocity he would have at the starting position r_0, and v' is its fall velocity. This relation clearly deviates from Einstein's addition law of velocities.

Special **polar coordinates** can be used to describe the exterior Schwarzschild solution. According to Kasner and Eisenhart it is generally accepted that one needs six dimensions to embed the Schwarzschild metric into a flat space. This raises the question of how a model of embedding class 2 can be linked to the interior solution, which is certainly of embedding class 1. We have shown in [4] that five dimensions are sufficient to embed the Schwarzschild exterior solution in a flat space.

We relate a pseudo-spherical coordinate system to a 5-dimensional Cartesian coordinate system $x^{a'}, a' = 0', 1', ...4'$ as given in Eq. (A6). A 4-dimensional hypersphere has the form $x_{a'}x^{a'} = X^2$ with radius X. If we shift the center of the sphere to $\bar{x}$, we have

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[2] This is like substituting the sides a and b of two different triangles into the Pythagorean theorem $a^2 + b^2 = c^2$, resulting in an ominous c.

$$(x_{a'} - \bar{x}_{a'})(x^{a'} - \bar{x}^{a'}) = X^2.$$

By suppressing all dimensions, except the extra dimension $x^{0'} = R$ and $x^{1'} = r$, we can unfold this equation into the two separate equations [4]

$$R^2 = 8M(r - 2M), \quad \bar{R}^2 = \frac{2}{M}\left(\frac{\bar{r}}{3} - 2M\right)^3. \tag{2.20}$$

These are the expressions for the Schwarzschild parabola (R, r) and Neil's parabola $(\bar{R}, \bar{r})$. The latter is the evolute of the Schwarzschild parabola and the Schwarzschild parabola is the associated evolvente.

Both curves are connected by the curvature vector of the Schwarzschild parabola. This vector is tangent to Neil's parabola and normal to Schwarzschild's parabola. While the tip of the curvature vector moves along the Schwarzschild parabola, the starting point moves along Neil's parabola. Both motions contribute to the structure of its model. These are calculated and illustrated in detail in [4]. The starting point of the curvature vector is called the pole. Thus, we have introduced a polar system with a pole that is not fixed but moves on Neil's parabola. This gives an additional degree of freedom, which might be the reason for the requirement for a sixth dimension. The polar coordinate system is depicted in Fig. A5. One of the parallel evolventes is the Schwarzschild parabola. With the help of (2.20), one can eliminate R and $\bar{R}$ and, thus, prepare the dimensional reduction. In addition, by using $\bar{r} = 3r$, one arrives at a 4-dimensional theory.

Since the Schwarzschild field can be geometrized, i.e., represented as a perceivable surface, we can use the curvatures of the slices of this surface to formulate the metric on this surface:

$$ds^2 = \rho_i \rho_k d\varepsilon^i d\varepsilon^k, \quad i = k \tag{2.21}$$

with the curvature radii and angles as follows:

$$\begin{gathered}\rho_1 = \rho = \sqrt{\frac{2r^3}{M}}, \quad \rho_2 = r, \quad \rho_3 = r\sin\vartheta, \quad \rho_4 = \rho\cos\varepsilon \\ \varepsilon^1 = \varepsilon, \quad \varepsilon^2 = \vartheta, \quad \varepsilon^3 = \varphi, \quad \varepsilon^4 = i\psi\end{gathered}. \tag{2.22}$$

The metric, written out in full, is as follows:

$$ds^2 = \rho^2 d\varepsilon^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta\, d\varphi^2 + \rho^2\cos^2\varepsilon\, di\psi^2. \tag{2.23}$$

More about this problem can be found in [4].

The 5-dimensional field equations consist of a set of subequations. These are the curvature equations of the slices of the surface and are of the type $\frac{d}{dr}\frac{1}{r} + \frac{1}{r^2} = 0$. In turn, if the surface is the basis of a model and one sets up the curvature equations of the slices, one can combine them to the Ricci. Finally, one obtains the Einstein tensor. By performing the dimensional reduction, one obtains Einstein's field equations and the associated metric without solving the differential equation given by the Einstein tensor.

Evidently, this special coordinate system cannot describe the region below $r = 2M$. The metric (2.23) is free of singularities but the field quantities derived from this metric have the aforementioned physical properties at the event horizon.

The metric of the **interior Schwarzschild solution**, with Schwarzschild's original form of the metric, is as follows:

$$ds^2 = \mathbb{R}^2 d\eta^2 + \mathbb{R}^2 \sin^2\eta\, d\vartheta^2 + \mathbb{R}^2 \sin^2\eta \sin^2\vartheta\, d\varphi^2 - \frac{1}{4}\left[3\cos\eta_g - \cos\eta\right]^2 dt^2 . \qquad (2.24)$$

Since the value of η is bounded, the spacelike part of the metric describes a cap of a sphere of radius $\mathbb{R}$. r_g and η_g are the values at the boundary to the exterior solution. Some authors have reformulated the original Schwarzschild metric by replacing variables in the line element with $\sin\eta = r/\mathbb{R}$, $\cos\eta = \sqrt{1-r^2/\mathbb{R}^2}$. They obtained the lapse function

$$\frac{1}{4}\left[3\sqrt{1-\frac{2Mr^2}{r_g^3}} - \sqrt{1-\frac{2M}{r}}\right]^2 . \qquad (2.25)$$

This notation may be more familiar to some authors, but it blocks the way to a geometric interpretation of the lapse function. However, we keep the Schwarzschild notation and replace idt by $\rho_g di\psi$. At the boundary $\eta = \eta_g$, the time-like part of the metric (2.24) becomes $\rho_g di\psi$ and coincides with the time-like part of the exterior solution. Thus, the 1st linking condition is satisfied. Application of Flamm's relation[3]

$$\rho = 2\mathbb{R} , \qquad (2.26)$$

one has $\rho_g di\psi = 2\mathbb{R} di\psi$ and finally, one obtains the time-like part of the metric

$$dx^4 = 3\mathbb{R}\cos\eta_g di\psi - \mathbb{R}\cos\eta\, di\psi , \qquad (2.27)$$

thus, the differences of two pseudo-circles[4] with radii $3\mathbb{R}\cos\eta_g$ and $\mathbb{R}\cos\eta$. The time corresponds to the growth of a pseudo-annulus sector.

Two linking conditions must be satisfied to match the interior and exterior solutions. The metrical coefficients and the tangents[5] (cutting tangents) must coincide at the boundary surface. We have shown this in detail in [21].

Schwarzschild also provides the pressure function. He states that the pressure at the center becomes infinite as the stellar object shrinks to

$$r_H = 2.25M > 2M. \qquad (2.28)$$

No star described by the Schwarzschild solution can be smaller than r_H. Thus, the inner horizon r_H covers the event horizon and degrades it to a mathematical artifact. This will be significant when we consider the collapse of a stellar object.

Black Holes are thought to be the final fate of collapsing stars. By redefining the variables, the exterior Schwarzschild solution is extended to the interior region. The matter is concentrated in a single point after a collapse. There, the matter has infinite density and space infinite curvature. This singular point is surrounded by empty space and shielded by the event horizon. But it is generally accepted that the exterior Schwarzschild field is

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3 Flamm's relation is little known in the literature, likely, because the paper was written in German.

4 Pseudocircles are open and have the same curvature everywhere, even at infinity

5 When we wrote our booklet [4], we were of the opinion that common tangents are generally used for the 2nd linking condition. Later on, we found in the literature linking conditions stemming from O'Brien and Synge, and from Lichnerowicz. These do not hold for the Schwarzschild model. We argue that they do not apply to models that can be geometrized.

generated by a stellar object of mass M. If one extends the exterior solution into the region, where the stellar object is located, then the question 'where has this object gone to' arises.

We recapitulate the properties of the *complete* Schwarzschild solution consisting of the interior and exterior solution [4]: The gravitational force inside the star is regular everywhere, and in the center of the star, it is zero. The interior solution always covers the event horizon of the exterior solution. Thus, the event horizon can never be experienced and all considerations, which refer to an exotic situation at this location, are obsolete. In particular, any attempt to extend the exterior Schwarzschild solution below the event horizon by a new choice of the coordinates or to allow a motion into this region is excluded by the structure of the model. Stars can collapse to ultra-massive objects. They can never shrink to a point singularity with infinitely high space curvature and mass density.

There are no unpleasant peculiarities in the complete Schwarzschild model. It satisfies all the prerequisites that one would expect from a field theory.

3. GRAVITATIONAL COLLAPSE

In the previous section, we already considered the collapse of a star. This problem has attracted great interest among researchers and many papers have appeared on the subject. In one of our papers [22] we gave an overview of the problem and added some remarks to some papers. In many of these articles, it was assumed that the collapsing object is surrounded by a Schwarzschild field, which is known to have an event horizon. As we have shown above, this event horizon cannot be reached and certainly not crossed. Nevertheless, many authors believe that the final state of a collapse is a singularity at $r = 0$. This means that a Black Hole can be created, which could be naked, i.e., the enclosing event horizon could be absent.

We also noticed that sometimes the stress-energy-momentum tensor is extended. This means that in addition to pressure and mass density, shear, viscosity, and heat flow are also taken into account. Einstein's field equations are then solved under these conditions. Not very surprisingly, some quantities or functions remain undetermined in the solution. One then tries to assign appropriate values to these quantities to give the model physical content. Occasionally, the solutions are made plausible with the help diagrams or tables. These solutions are not exact solutions of Einstein's field equations – although it is sometimes claimed that they are exact. By assuming additional quantities in the stress-energy-momentum tensor, the field equations turn out to be underdetermining. We believe that Nature does not favor such solutions.

The correct procedure would be to compute the Einstein tensor from a metric, and then compute the stress-energy-momentum tensor. In the case of the above-mentioned methods, however, the opposite approach was used. One specifies the right-hand side of Einstein's field equations and then sees what happens with the left-hand side, taking into account the applied metric. Since the reverse approach is used here, we call such models *retro models*.

Some authors start with a *collapsing metric*. This might be a fuzzy communication for specifying a metric to describe a collapse. But if 'collapsing metric' is taken literally, i.e., if one assumes a structure in this metric that explains a collapse, then the knowledge of Gauss is ignored: A line element can only describe the curvature of a surface but not the change of the curvature. For the latter, we need another system of differential equations, i.e., the Bianchi identities. The resulting conservation laws lead to a relation that defines the time-dependence of the metrical quantities. To describe a collapse that does not violate the laws of relativity and does not lead to any singularity, we turn the clock back to 1916. Knowledge of three papers, i.e., the two by Schwarzschild and the one by Flamm [23], is sufficient for this task. Furthermore, knowledge of the papers of Ricci, Bianchi, and Levi Civita, which were written around 1900, is advantageous. The authors established the tetrad calculus, which provides us with rods and clocks to measure spacetime. The tetrads can also be used to avoid coordinate effects[6] that cannot be interpreted physically.

As mentioned above, we do not assume a collapsing metric, but a static one. In doing so, we allow the possibility that a parameter can be ascribed to be time-dependent. Since the practicality of the exterior Schwarzschild field has been confirmed by experimental results, it is obvious to use the interior Schwarzschild solution for the collapse. Both the solutions, the interior, and the exterior, can be geometrized, i.e., can be explained by surfaces. The interior surface is a spherical cap with the radius $\mathcal{R}$, which nestles against the Flamm paraboloid of the exterior solution.

We set up the radius of the spherical cap as a function of time: $\mathcal{R} = \mathcal{R}(t)$, $\mathcal{R}$ decreases with time. Since the two linking conditions should still hold, the collapse occurs in such a way that the spherical cap slides down on Flamm's paraboloid. We have illustrated this in Fig.1

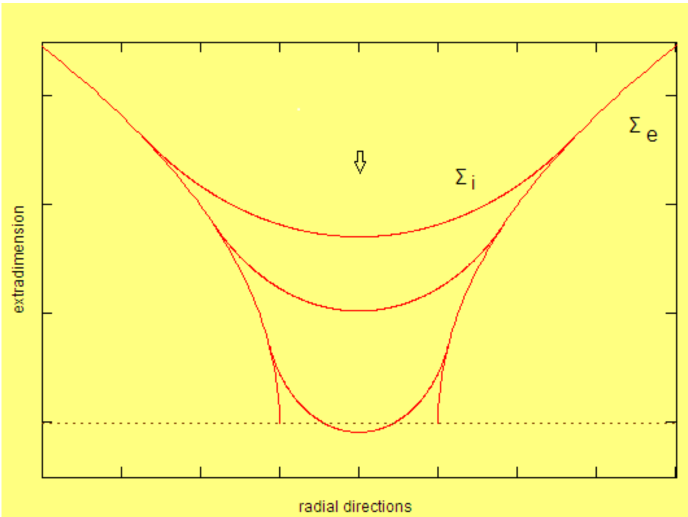

Fig. 1. Collapse of Schwarzschild interior solution

Further, we remember Flamm's relation $\rho = 2\mathcal{R}$. Regarding the linking boundary, this means that the radius of curvature ρ of Flamm's paraboloid determines the radius of the spherical cap at any time during the collapse. Thus, the collapse of the interior solution is closely linked to the exterior one. According to Birkhoff's theorem, the exterior solution, i.e., Flamm's paraboloid remains unchanged during the collapse. Thus, the collapse is described by a family of self-similar static solutions, the Schwarzschild interior solutions. We described this in detail in our paper [26]. There, we also defined the collapse velocity, which allows switching between comoving and non-comoving reference systems using a Lorentz transformation. Due to the shrinking of the stellar object, the mass density

[6] In the paper [24], we showed simple examples of how the use of Ricci-rotation coefficients and Christoffel symbols leads to different results.

increases, and the volume elements decrease. The latter lead to tidal forces acting in addition to gravity.

By integrating the collapse velocity, we calculated the proper time needed for the stellar object's surface to reach a given position during the collapse. The integral diverges at the inner horizon $r_H = 2.25M$. This means that this position can only be reached asymptotically after an infinite amount of time. A plot of the time function shows a similar behavior as shown in the Appendix with Fig. A1. This means that the collapse does not end in a singularity, but that a massive, or possibly, an ultra-massive object is formed. Mitra introduced the term ECO (Eternally Collapsing Objects) for such objects in a series of papers and a book[7] [26].

Mitra illuminated the problem from an astrophysical point of view. Among other insights, he found that so-called Black Holes have magnetic fields. He called these objects MECOs (Magnetic Eternally Collapsing Objects). Also, Black Holes cannot be the source of the sometimes observed X-ray bursts.

Since the Schwarzschild theory definitely excludes Black Holes, such objects could only be preserved by turning to other models. However, such a task would cause considerable difficulties. First, one will not be able to discard the exterior Schwarzschild solution. According to Birkhoff's theorem, the following applies: *A spherically symmetric field in empty space must be static, with a metric given by the Schwarzschild solution*. The question of whether the interior solution can be replaced by another friendlier one concerning Black Holes remains. Such a model would have to be formally suitable for describing a collapse and satisfy the two linking conditions. Since the boundaries of the exterior Schwarzschild solution consist of circles, the alternative model must also be bordered by circles. To describe the stellar object as a compact geometric object, the boundary circles must be connected by arcs, which must have common tangents with Flamm's paraboloid. This inevitably requires that these arcs must in turn have circular shape. This leads to the conclusion that the geometric object must be a spherical cap. Moreover, the interior Schwarzschild solution could also have been found, if one had assumed a suitable surface and then determined the corresponding metric.

4. COSMOLOGICAL PROBLEMS

In this section, we discuss some specific problems in cosmology. In particular, we discuss the use of coordinate systems and some difficulties that arise from them. It is widely accepted that our universe expands in free fall. When discussing a specific model, it is convenient to start with a metric in **comoving spherical coordinates**. The standard form is as follows:

$$ds^2 = \mathcal{K}^2 \left[\frac{1}{1 - k\frac{r'^2}{\mathcal{R}^2}} dr'^2 + r'^2 d\vartheta^2 + r'^2 \sin^2\vartheta\, d\varphi^2 \right] - dt'^2 . \tag{4.1}$$

[7] In an early draft (private communication), he inquired about researchers who denied the existence of Black Holes. Some journals rejected their papers, online contributions were deleted, and some authors had to self-deny in order continuing their publishing. One would think that in science only arguments count, and not the preconceived notions.

Here r' is the comoving radial coordinate; $\mathcal{K}$ the scale factor; $\mathcal{R} = \text{const.}$ the radius of a hypersphere; and k the curvature parameter. A large class of cosmological models is based on this metric. They are called FRW models. For the curvature parameter k, one admits the values $k = \{1, 0, -1\}$, denoting a positively curved, flat, and negatively curved space, respectively. The latter ones have an infinite extension. The coordinate time t' is the universal time for all observers. It is also the proper time for comoving observers. Here, a contradiction arises. According to Lemaître a freely falling observer has universal time and *local flatness*, i.e., $k = 0$. Thus, models having $k = 1$ or $k = -1$ are ruled out. We called these models hybrid models in [27]. Local flatness should not be confused with *global flatness*. Lemaître showed that the metric of a positively curved space with $k = 1$ in **non-comoving coordinates** changes to a metric with $k = 0$ using comoving coordinates. Thus, k is altered by a coordinate transformation and is not an invariant quantity that describes the curvature of space. We prefer to call k as the *form parameter* of the metric. Local flatness means that freely falling observers do not experience any gravitational forces. This is Einstein's elevator principle, which we discussed in several papers [8,28,29,30].

Solving Einstein's field equations and the conservation law for the mass density, respectively, one obtains the Friedman equation. Since the original Friedman model is pressureless, one tried to improve the situation by manually inserting pressure into the right-hand side of the field equations. Of course, one did not get an exact solution to Einstein's field equations. Such models we called retro models. Not very surprisingly, the field equations contain several undefined variables, the Ωs and the deceleration parameter. This conflicts with the requirement that the universe is in free fall and that there should be no acceleration according to Einstein's elevator principle. Although the FRW models are hybrid and retro, they are called standard models. Astrophysicists are trying hard to equip the Ωs with physical meaning and to manipulate their values to fit observational data. Moreover, the Friedman equation, describing the expansion of the cosmos, contains an acceleration term and is quite complex.

In contrast, it is very easy to construct a cosmological model that includes pressure and is an exact solution to Einstein's field equations. The dS cosmos is well known in the literature. It is a model with $k = 1$, thus positively curved. It can be represented geometrically by a pseudo-hypersphere with constant radius $\mathcal{R}$. It contains forces that drive particles apart in all directions. Since the dS cosmos is homogeneous, the particles would move away from every point in every direction. This scenario does not seem to be quite physical.

The problem can be solved immediately by dropping the condition $\mathcal{R} = \text{const.}$. The cosmos is expanding and de Sitter's forces are those forces that pull the mass particles apart due to the increase in the volume elements.

The model is as simple as possible. It is based on a pseudo-hypersphere and the Friedman equation has the form[8]

$$\dot{\mathcal{R}} = 1, \quad \ddot{\mathcal{R}} = 0. \tag{4.2}$$

The EOS is

$$\mu_0 - 3p = 0. \tag{4.3}$$

---

[8] The overdot denotes differentiation with respect to the universal time.

We called our model the Subluminal Model [31]. It is linear in expansion: no acceleration of the expansion can be deduced from this model, contrary to the claims of Perlmutter and Riess, who based their ideas on an FRW model. Moreover, there are no superluminal velocities of receding galaxies.

Hubble's law, with v as the recession velocity of the galaxies and H as the Hubble parameter

$$v = Hr\,, \tag{4.4}$$

admits velocities higher than light if the distance r is unbounded. This applies to universes with $k \neq 0$ in (4.1). This would have the consequence that galaxies cannot exchange information through light signals. Galactic island formation would occur. For a closed universe we have $r = \mathbb{R}\sin\eta$, where η is the polar angle. The highest value of r is $r = \mathbb{R}$, denoting the equator of the pseudo-hypersphere to an observer at the pole. It is the cosmic horizon. Due to the high recession velocity of the galaxies at the cosmic horizon, the wavelength of the light is shifted so far into the red that it can no longer be seen. Thus, the universe beyond the cosmic horizon cannot be seen by an observer at the pole. Any arbitrary observer can define his position as a pole on the pseudo-hypersphere and has his own horizon. He can observe other regions of the cosmos and send information to distant observers. Thus, any observer can get information about parts of the sky that he cannot observe.

After developing the Subluminal Model, we found a model, published by Melia[9] and his coworkers in several papers. Surprisingly, the final results of our model agreed with those of Melia, although both models start with different methods. Melia's R_h=ct model is based on a flat, infinite space with an infinite amount of matter being created by the Big Bang. Based on a flat universe, he created a cosmologic horizon by comparing it to the event horizon of the Schwarzschild theory. Melia referred to Weyl's cosmological principle and to Birkhoff's theorem. An enclosed mass $M = M(r_h)$ of a given volume in the universe determines the Hubble radius $r_h = 2GM/c^2$, which leads to the relation $R_h = ct$. Hubble's radius is defined as the distance that light has traveled since the Big Bang; t represents the age of the universe and r_h is the location at which the expansion rate reaches the velocity of light.

Melia starts with a metric in comoving coordinates with $k = 0$ and concludes that the space is globally flat. Taking into account Lemaître's results and Einstein's elevator principle, one can reformulate Melia's model in such a way as to see that his model is only locally flat and thus identical to our Subluminal Model. We have worked out this in a few articles [24,28,30,32,33]. The main problem in cosmology is that little attention is paid to the physical and geometrical consequences[10] of expansion in free fall.

Melia, who has a large collection of astrophysical data at his disposal, has continuously evaluated them and found that they tend to favor a linear model with constant expansion. He used this data to support his R_h=ct model and in doing so also provided arguments for our Subluminal Model. In one of his papers [34] he mentioned 10 paradoxes and inconsistencies related to 27 different kinds of observations concerning FRW models.

[9] Most of the papers by Melia and colleagues are listed in [31].

[10] Gravitational physicists do not have access to astrophysical data and usually do not have the expertise to interpret them. Cosmologists are not born with differential geometry. There seems to be some misunderstanding and no willingness for interdisciplinary cooperation.

In another paper [35] he based his arguments on the observations of the PLANCK project. He showed with a graph (Fig. 2) that the results are consistent with the linear model and not at all consistent with the FRW region (the blue area of the graph).

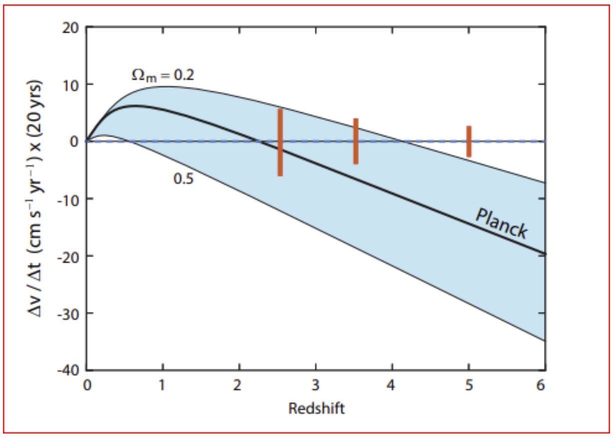

Fig 2. Depiction of the data of the PLANCK project by Melia

After discussing a cosmological model in comoving coordinates, several authors try to present the theory in non-comoving coordinates, though with little success. They call such coordinates **Schwarzschild** or **curvature coordinates**. We show that in general such coordinates do not exist. Again, we start with the static dS cosmos and its particles moving in free fall. Using tetrad calculus, we find that the acceleration force is in the radial direction and has only one component U_1. Performing a Lorentz transformation on a system comoving with these particles, one obtains with the inhomogeneous transformation law of the Ricci-rotation coefficients

$$U_m = \{U_1, 0, 0, 0\} \quad \rightarrow \quad {}'U_{m'} = \{0, 0, 0, {}'U_{4'}\}. \tag{4.5}$$

Here, ${}'U_{4'} = -i/\mathcal{R}$, where $\mathcal{R} = \text{const.}$, is the radial tidal force expanding the particles' swarm. Abandoning the condition $\mathcal{R} = \text{const.}$, we obtain our Subluminal Model. Using comoving coordinates, we obtain a quantity like ${}'U_{m'}$ mentioned in (4.5). It describes the expansion of a volume element in the radial direction. Now, we face a transformation to a non-comoving system. Bearing in mind that the expansion of a volume element is a physical fact, it is experienced not only by a comoving observer but by any observer, including non-comoving observers. Using the recession velocities of the galaxies for a Lorentz transformation, we perform a transformation to a non-comoving reference system. With the inhomogeneous transformation law of the Ricci-rotation coefficients we obtain

$$ {}'U_{m'} = \{0, 0, 0, {}'U_{4'}\} \quad \rightarrow \quad U_m = \{U_1 + f_1, 0, 0, f_4\}. \tag{4.6}$$

Here, U_1 is the dS acceleration and $f_m = \{f_1, 0, 0, f_4\}$ are the contributions of the tidal forces, seen by the non-comoving observers. In general, $U_1 + f_1$ is not a gradient of a space-dependent function. Thus, no lapse function can be derived from this expression, and no coordinate system can be associated with the non-comoving reference system. The lapse function of the dS model can be obtained only by turning off the expansion with $f_m = 0$.

Thus, Schwarzschild coordinates and curvature coordinates do not exist for linear models and we argue that they also do not exist for FRW models either. This is suggested to us by unsuccessful attempts by several authors.

5. CONCLUSIONS

We showed that the search for new coordinates to describe the inner region of the exterior Schwarzschild solution is a hopeless undertaking. Coordinates can never affect the geometrical and physical structure of a model. Thus, the only possibility to cover the empty inner region of the exterior solution is to fill it with the interior Schwarzschild solution. The complete Schwarzschild solution definitely excludes the possibility of Black Holes. Once a massive star has collapsed, an ECO emerges. ECOs are most likely the dark objects observed in the universe.

Coordinates are also the subject of discussion in cosmology. Commonly used are transformations from comoving coordinates to non-comoving coordinates and vice versa. In doing so, the curvature parameter k changes its value. Therefore, the usual symbolization for k is of little use. The universe does not change its structure with a coordinate transformation. In particular, we have pointed out the difference between local and global flatness. The confusion of these concepts leads to very different interpretations of cosmological models.

If our arguments are correct, Nature is rather simple. Black Holes do not exist; there are no singularities either, whether naked or dressed. The Hawking radiation, no hair theorems, cosmic censorship, and the quantum mechanical information paradox are obsolete.

Further, we live in a linear expanding universe, a universe that expands without acceleration. It is described by a pseudo-hypersphere that can be understood with basic mathematical methods. The Friedman equation is quite simple. The recession velocities of the galaxies are subluminal, and the laws of Special Relativity are respected.

6. MATHEMATICAL APPENDIX

- Tetrad systems of the Gullstrand metric:

$$\text{(A)}\quad \overset{1}{e}_1=\alpha,\quad \overset{2}{e}_2=r,\quad \overset{3}{e}_3=r\sin\vartheta,\quad \overset{4}{e}_1=-i\alpha v,\quad \overset{4}{e}_4=a$$
$$\underset{1}{e}^1=a,\quad \underset{2}{e}^2=\frac{1}{r},\quad \underset{3}{e}^3=\frac{1}{r\sin\vartheta},\quad \underset{1}{e}^4=i\alpha v,\quad \underset{4}{e}^4=\alpha\,, \tag{A1}$$

$$\text{(B)}\quad \overset{1}{e}_1=1,\quad \overset{2}{e}_2=r,\quad \overset{3}{e}_3=r\sin\vartheta,\quad \overset{1}{e}_4=-iv,\quad \overset{4}{e}_4=1$$
$$\underset{1}{e}^1=1,\quad \underset{2}{e}^2=\frac{1}{r},\quad \underset{3}{e}^3=\frac{1}{r\sin\vartheta},\quad \underset{4}{e}^1=iv,\quad \underset{4}{e}^4=1\,. \tag{A2}$$

- Freely falling observers in Einstein-Rosen coordinates:

  One has

$$dT'=\frac{R_0}{\sqrt{R_0^2+16M^2}}\,\frac{1}{\dfrac{4M}{\sqrt{R_0^2+16M^2}}-\dfrac{4M}{\sqrt{R^2+16M^2}}}\,\frac{R}{4M}\,dR\,.$$

  After some rearrangement, one obtains an integral of the type

$$\int \frac{\sqrt{x}}{\sqrt{x}-1}dx = x + 2\sqrt{x} + 2\ln(1-\sqrt{x}), \quad x<1, \quad \lim_{x\to 1}\int \frac{\sqrt{x}}{\sqrt{x}-1}dx = \infty .$$

For $x = \frac{R^2+16M^2}{R_0^2+16M^2}$, one gets the rise time $f(R)$. Starting from $R=0$ and ending at R_0, it increases to infinity. In this case, it is easy to mirror the function so that an observer starting at R_0 reaches the event horizon at $R=0$ after an infinitely long time. Replacing the variable R by $R_0 - R$ in the time function, the observer starts at $R=R_0$ and passes through the fall distance at $R=0$. We have

$$f(R,R_0) = \frac{R_0}{32M^2}\left(R_0^2+16M^2\right)\times$$

$$\times\left[\frac{(R_0-R)^2+16M^2}{R_0^2+16M^2} + 2\sqrt{\frac{(R_0-R)^2+16M^2}{R_0^2+16M^2}} + 2\ln\left(1-\sqrt{\frac{(R_0-R)^2+16M^2}{R_0^2+16M^2}}\right)\right]+C \qquad \text{A(3)}$$

After a suitable choice of the integration constant

$$T'(R,R_0) = f(R,R_0) - f(R_0,R_0),$$

we obtain the following graph

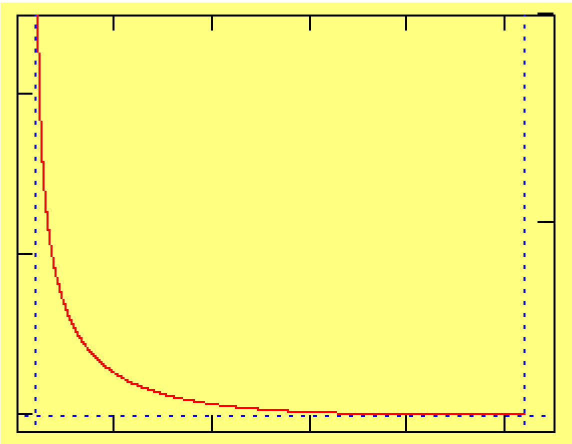

Fig. A1. The fall time near the event horizon

In Fig. A1, the left vertical line marks the event horizon. It turns out that no object infalling from a finite or infinite position can reach the event horizon in finite proper time.

- Isotropic coordinates:

  One should be aware that both the values $\bar{r}=0$ and $\bar{r}=\infty$ denote infinity.

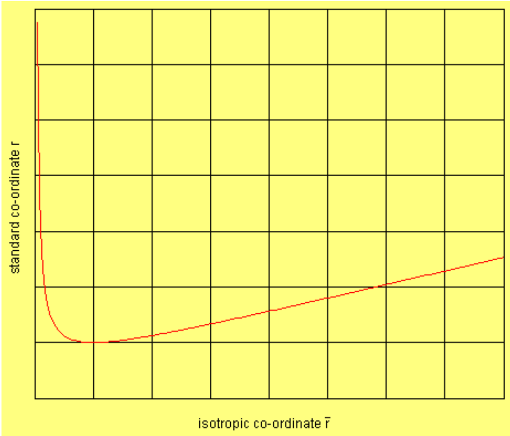

Fig. A2. Isotropic vs standard coordinates

- Eddington and Finkelstein coordinates:

  4-beine

$$\overset{1}{e}_1 = \alpha, \quad \overset{4}{e}_1 = -i\alpha v^2, \quad \overset{4}{e}_4 = a$$
$$\underset{1}{e}^1 = a, \quad \underset{1}{e}^4 = \ i\alpha v^2, \quad \underset{4}{e}^4 = \alpha \tag{A4}$$

  Transformation matrix

$$\Lambda_1^{1'} = 1, \quad \Lambda_4^{1'} = 0, \quad \Lambda_1^{4'} = -i\alpha^2 v^2, \quad \Lambda_4^{4'} = 1$$
$$\Lambda_{1'}^{1} = 1, \quad \Lambda_{4'}^{1} = 0, \quad \Lambda_{1'}^{4} = \ i\alpha^2 v^2, \quad \Lambda_{4'}^{4} = 1 \tag{A5}$$

- Kruskal metric:

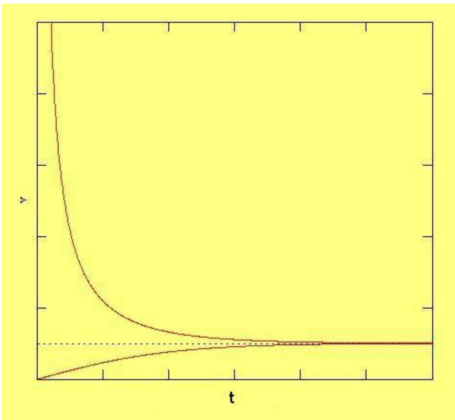

Fig. A3. The velocities if the Kruskal metric

- Isothermal coordinates:

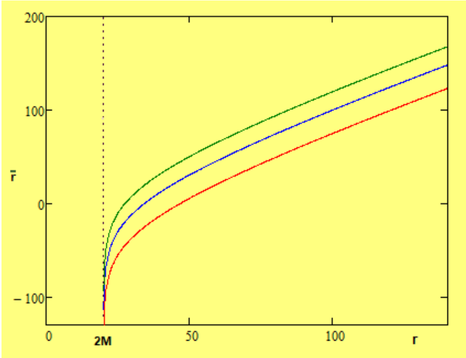

Fig. A4. Isothermal coordinates vs Schwarzschild coordinates

The three curves correspond to different constants.

- Polar coordinate system of the Schwarzschild exterior solution:

Embedding:

$$\begin{aligned}
X^{3'} &= X\sin\varepsilon\sin\vartheta\sin\varphi \\
X^{2'} &= X\sin\varepsilon\sin\vartheta\cos\varphi \\
X^{1'} &= X\sin\varepsilon\cos\vartheta \\
X^{0'} &= X\cos\varepsilon\cos i\psi \\
X^{4'} &= X\cos\varepsilon\sin i\psi
\end{aligned} \tag{A6}$$

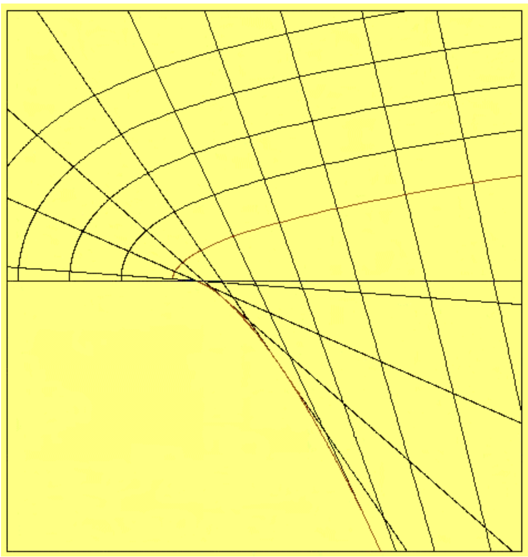

Fig. A5. The polar coordinate system of the Schwarzschild exterior solution

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