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# KERR GEOMETRY VI. THE SURFACES

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Contents



Abstract: An embedding in a five-dimensional flat space for the Kerr theory will be derived. An elliptically squashed surface, similar to Flamm's paraboloid, endowed with an anholonomic structure provides a 2-metric, the first part of the Kerr metric. An additional axisymmetric surface and a double surface explain the Kerr line element.

## 1. INTRODUCTION

It was Einstein's vision to explain the effects of gravity by means of the geometry of a four-dimensional surface which is embedded in a higher dimensional flat space. Nevertheless, decades after the installation of the general theory of relativity this draft has been realized only in few models. The reason might lie in the fact that this draft has been formulated too narrowly.

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Cross terms of a metric are to be explained not by properties of a surface, but by a structure on a surface. Before one looks for a surface, one should remove these cross terms by a suitable anholonomic transformation. Some models may require still another anholonomic structure. By utilizing a double surface we are able to keep low the number of the required embedding dimensions. We will show that all these requirements are necessary to embed the Kerr geometry in a five-dimensional flat space.

In Sec. 2 we will show how to obtain a surface for the spatial part of the Kerr geometry and we will give reasons for the concept of the *physical surface*.

In Sec. 3 we will deal with the curvature of the Kerr surface and we will show that the Kerr line element is the line element of this physical surface.

# 2. THE SURFACES

At first we treat the spatial part of the Kerr metric. For $\vartheta = \pi/2$ Sharp [1] has found a surface whose parallels are circles. Our ansatz differs from that of Sharp by the fact that we have transformed away the rotational contributions.

The Kerr geometry is built up on a base which is parameterized by the use of the Boyer-Lindquist co-ordinate system with its confocal ellipses and hyperbolae. Hence, we suppose that the $[r,\vartheta]$-slice will be an elliptically squashed surface. If the rotation parameter a (the eccentricity of the ellipses) vanishes, this surface coincides with Flamm's paraboloid, the geometrical interpretation of the $[r,\vartheta]$-slice of the Schwarzschild geometry.

The cross terms of a metric, as they appear, for instance, in a rotating model, indicate an additional structure *on* a surface rather than a property *of* the surface. We have pointed out [2,3] that the exterior as well as the interior Schwarzschild solution can be embedded in a five-dimensional flat space, on condition that one uses a double surface. Because the Kerr metric is closely related to the Schwarzschild metric, we will try to apply the same strategy to this model.

By using the Boyer-Lindquist co-ordinate system the line element of the Kerr metric can be written as

$$ds^2 = {dx^1}^2 + {dx^2}^2 + \left[\alpha_R dx^3 + i\alpha_R \omega\sigma dx^4\right]^2 + a_S^2\left[-i\alpha_R\omega\sigma dx^3 + \alpha_R dx^4\right]^2 \qquad (2.1)$$

wherein

$$dx^1 = \alpha_S a_R dr,\ dx^2 = \Lambda d\vartheta,\ dx^3 = \sigma d\varphi,\ dx^4 = idt$$
$$\alpha_R = \frac{A}{\Lambda},\quad a_R = \frac{\Lambda}{A}, \omega = \frac{a}{A^2},\quad \sigma = A\sin\vartheta,\quad A^2 = r^2 + a^2,\ \Lambda^2 = r^2 + a^2\cos^2\vartheta \,. \qquad (2.2)$$
$$a_S = \frac{\delta}{A},\quad \alpha_S = \frac{A}{\delta},\quad \delta^2 = r^2 + a^2 - 2Mr$$

A and r are the semi-axes of confocal ellipses with eccentricity a, ω the observer's angular velocity, σ the observer's distance from the rotation axis and $\alpha_R$ the Lorentz factor of this rotation.

In former papers [2,5] we have discussed the field strengths and field equations of the metric (2.1) and in subsequent papers [6,7] we have shown that it is possible to correlate with the components of the field strengths additional components in an extra dimension. In this paper we will search for a surface matching these components.

If we perform a non-Lorentzian transformation

$$\overset{m'}{e}_i = \Xi^{m'}_{m}\overset{m}{e}_i$$

acting on the 4-bein of the metric (2.1) with the transformation coefficients

$$\begin{aligned}
&\Xi^3_{3'} = \alpha_R, \quad \Xi^3_{4'} = \ i\alpha_S\alpha_R\omega\sigma, \quad \Xi^4_{3'} = -ia_S\alpha_R\omega\sigma, \quad \Xi^4_{4'} = \alpha_R \\
&\Xi^{3'}_3 = \alpha_R, \quad \Xi^{3'}_4 = -i\alpha_S\alpha_R\omega\sigma, \quad \Xi^{4'}_3 = \ ia_S\alpha_R\omega\sigma, \quad \Xi^{4'}_4 = \alpha_R
\end{aligned} \tag{2.3}$$

we obtain a static model [4] which exhibits most of the geometric properties of the Kerr metric, but does not take into account the rotational structure on the surface. This line element of this model reads as

$$ds^2 = \alpha_S^2 a_R^2 dr^2 + \Lambda^2 d\vartheta^2 + \sigma^2 d\varphi^2 + a_S^2 dx^{4^2}, \quad dx^4 = idt \tag{2.4}$$

The radial line element provides us with two pieces of information. The first one concerns the elliptical/hyperbolical structure of the theory. Firstly, we note, that the radial distance of two neighboring ellipses in the basis plane is $a_R dr$, where dr is the increase of the minor semi axes of the ellipses. $a_R dr$ reduces to dr at the minor semi axes. Secondly, we interpret $\alpha_S$ as $1/\cos\varepsilon$, where $\varepsilon$ is the angle of ascent at the minor semi axis of one of the confocal ellipses. Its orientation is taken to be cw for the sake of physical interpretation. With the help of (2.2) we calculate the ascent of the surface at the minor semi axis as

$$\tan\varepsilon = -\frac{\sqrt{2Mr}}{\delta}. \tag{2.5}$$

The solution of the integral

$$x^{0'}(r_1) = -\int_{r_0}^{r^1} \tan\varepsilon dr, \quad r_0 = M + \sqrt{M^2 - a^2} \tag{2.6}$$

has no closed form. $r_0$ is the boundary of the geometry, most commonly called the event horizon. If we use

$$\begin{aligned}
x^{0'} &= -\int \tan\varepsilon\, dr \\
x^{1'} &= r\cos\vartheta \\
x^{2'} &= A\sin\vartheta\cos\varphi \\
x^{3'} &= A\sin\vartheta\sin\varphi
\end{aligned} \tag{2.7}$$

as rectilinear co-ordinates for a point on the surface, we obtain for the $[r,\vartheta]$-slice an elliptically squashed surface as shown in Fig.1. For the $[r,\varphi]$-slice we get a similar surface but with circles as parallels, their radii being $A\sin\vartheta$. Unfortunately, the metric of these surfaces is not the Kerr metric.

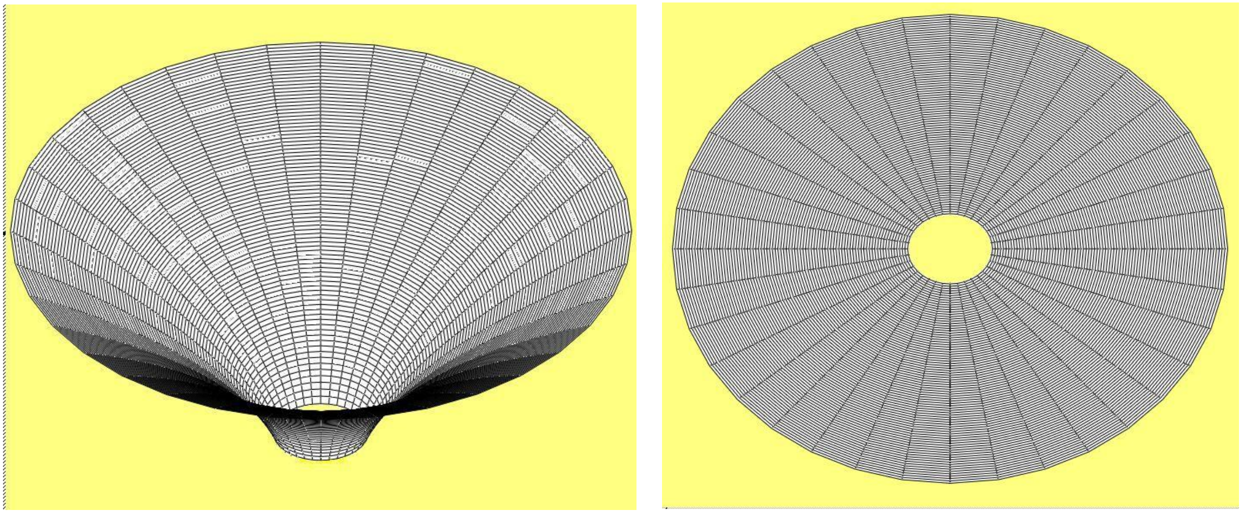

Fig.1.

As the distance of two neighboring ellipses is $a_R dr$, the ascent of the elliptically squashed surface is a function of r and $\vartheta$. From (2.7) we get

$$\begin{gathered} dx^{0'}_{holonomic} = -\tan\varepsilon\, dr = -\tan\varepsilon\, \alpha_R dx^{\underline{1}} = -\tan\alpha\, dx^{\underline{1}} \\ \tan\alpha = \tan\varepsilon\, \alpha_R(r,\vartheta), \quad dx^{\underline{1}} = a_R dr \end{gathered} \quad . \tag{2.8}$$

To get the Kerr metric one has to introduce an anholonomic structure on this surface by rotating the radial tangent vector and the normal vector of the surface through an angle $\alpha-\varepsilon$ about the axis of the binormal vector. If we demand that the projection of the holonomic and anholonomic radial line element onto the basic plane has the same value, we get a simple construction of how to cut off the radial line element from the anholonomic hyperplanes. We obtain the non-integrable function

$$dx^{0'}_{anholonomic} = -\tan\varepsilon\; a_R(r,\vartheta)\, dr = -\tan\varepsilon\, dx^{\underline{1}} \cdot \tag{2.9}$$

Thus, all anholonomic radial vectors have the same ascent at a parallel of the surface. The family of all these anholonomic hyperplanes is part of the physical surface, the world we are living in. The physical surface is the place of all possible observations and measurements. If we constrain the higher-dimensional line element to the physical surface we get from

$$ds^2 = dx^{0'^2} + dx^{1'^2} + dx^{2'^2} + dx^{3'^2}$$

and from the last three lines of (2.7) and from (2.9)

$$ds^2 = \left(\tan^2\varepsilon + 1\right)\frac{\Lambda^2}{A^2} dr^2 + \Lambda^2 d\vartheta^2 + A^2 \sin^2\vartheta\, d\varphi^2$$

or

$$ds^2 = \alpha_S^2 a_R^2 dr^2 + \Lambda^2 d\vartheta^2 + \sigma^2 d\varphi^2 \tag{2.10}$$

as the spatial part of the Kerr metric. In the next section it remains to derive the time-like part of the line element. We note for the components of the normal vector, the tangent vector, and the binormal vector

$$\begin{aligned} n^{a'} &= \{\cos\varepsilon, \quad \sin\varepsilon\cos\theta, \sin\varepsilon\sin\theta\cos\varphi, \ \sin\varepsilon\sin\theta\sin\varphi\} \\ m^{a'} &= \{-\sin\varepsilon, \cos\varepsilon\cos\theta, \cos\varepsilon\sin\theta\cos\varphi, \cos\varepsilon\sin\theta\sin\varphi\}, \\ b^{a'} &= \{\quad 0 \quad, \quad -\sin\theta, \quad \cos\theta\cos\varphi, \quad \cos\theta\sin\varphi\} \end{aligned} \tag{2.11}$$

where $a' = \{0',1',2',3'\}$ and θ are the off-axis angles of the curvature vectors of the ellipses, related to $\vartheta$ by

$$\sin\theta = \frac{r}{\Lambda}\sin\vartheta, \quad \cos\theta = \frac{A}{\Lambda}\cos\vartheta . \tag{2.12}$$

To get more information on the geometry we will apply the results of the theory of surfaces to this model. One has to bear in mind that the Kerr geometry has anholonomic features. As a consequence of this, asymmetries will appear in some expressions, which are not known from the ordinary surface theory.

# 3. CURVATURES

In former papers [2,5,6,7] we have shown that the curvatures of the surface and the curvatures of their slices play an important role in the definition of the field strengths. The curvature vector of the elliptical parallels is

$$\rho_E = \frac{\Lambda^3}{rA} \tag{3.1}$$

and of the hyperbolical orthogonal trajectories

$$\rho_H = -\frac{\Lambda^3}{a^2\sin\vartheta\cos\vartheta} \tag{3.2}$$

and the curvature vector of a $[r,\varphi]$ -slice simply is the radius $\sigma = A\sin\vartheta$ of a circle. The curvature of the radial lines of the surface is calculated at the minor semi axes of the ellipses with the help of (2.9) as

$$\rho = A\sqrt{\frac{2r}{M}}\Phi^2, \quad \Phi^2 = \frac{r^2+a^2}{r^2-a^2} . \tag{3.3}$$

If we set a=0, this expression reduces to the curvature of the Schwarzschild parabola $\rho=\sqrt{\frac{2r^3}{M}}$. As the fundamental quantities of the physical surface are contracted by the elliptical factor $a_R$ on their way from the minor semi axis to the major semi axis, the curvature vector of the physical surface has the form

$$\rho_S=\rho a_R=\Lambda\sqrt{\frac{2r}{M}}\Phi^2 . \tag{3.4}$$

Since (3.3) can be written as

$$\rho=-2r\Phi^2\frac{1}{\sin\varepsilon},\quad v_S=\sin\varepsilon=-\frac{r}{A}\sqrt{\frac{2M}{r}}, \tag{3.5}$$

where $v_S$ is the velocity of a freely falling observer (also free from dragging effects) we are able to calculate the co-ordinates of the points of the evolutes of the radial lines of the holonomic surface:

$$\bar{r}=r-\rho\sin\varepsilon=r\left(1+2\Phi^2\right)\cdot \tag{3.6}$$

With the help of

$$d\bar{r}=\left[1+2\Phi^2+\frac{4r^2}{r^2-a^2}\left(1-\Phi^2\right)\right]dr$$

we can show that

$$\frac{d\rho}{d\bar{r}}=-\frac{1}{v_S}. \tag{3.7}$$

If the tip of the curvature vector ρ is moved on a radial line of the surface, its tail will move on the evolute and cover a distance of dρ. If we move the tip of the vector ρ on all elliptical slices of the holonomic surface in such a manner that the ascent of this vector is constant, we obtain a surface of revolution which has to be endowed with the same nonholonomicity as discussed in Sec. 2. Denoting the local components of the displacement on the evolute by $d\bar{x}^a$, we find

$$d\bar{x}^0=-d\rho,\quad d\bar{x}^1=0 . \tag{3.8}$$

The latter relation is the embedding condition for a surface generated by the evolute. The transition from the global rectilinear co-ordinate system to the local reference system is achieved by

$$d\bar{x}^{0'}=\cos\varepsilon\, d\bar{x}^0-\sin\varepsilon\, d\bar{x}^1,\quad d\bar{x}^{1'}=\sin\varepsilon\, d\bar{x}^0+\cos\varepsilon\, d\bar{x}^1 \cdot$$

Using (3.8) we obtain

$$d\bar{x}^{0'}=\cot\varepsilon\, d\bar{r},\quad d\bar{x}^0=\frac{1}{\sin\varepsilon}d\bar{x}^{1'} \tag{3.9}$$

for the semi minor axis, while at any other positions on the ellipses one has

$$\begin{aligned} d\bar{x}^{0'}_{holonomic} &= \cot\varepsilon\,\alpha_R d\bar{x}^{\underline{1}} \\ d\bar{x}^{0'}_{anholonomic} &= \cot\varepsilon\, a_R d\bar{r} = \cot\varepsilon\, d\bar{x}^{\underline{1}} \cdot \end{aligned} \tag{3.10}$$

For the Schwarzschild case (a = 0) we obtain from (3.6) for the evolute

$$\bar{r} = 3r, \quad \bar{x}^{0'2} = \frac{2}{M}\left(\frac{\bar{r}}{3} - 2M\right)^3,$$

the equation for Neil's parabola.

By drawing lines in the direction of ρ we are able to set up an all-over-space co-ordinate system by using these lines, the integral lines containing the points (2.6), and all parallel lines to these integral lines. Then also we are able to perform a prolongation of all quantities of the surface normal to this surface. This enables us to calculate the change of the quantities in the local 0-direction. If $\delta x^0$ is the distance of two neighboring integral lines of the parallel surfaces one has similar to (3.8) and (3.10)

$$\delta x^0 = d\rho, \quad dx^1 = 0, \quad dx^{0'}_{anholonomic} = \cot\varepsilon\, dx^{\underline{1}}, \tag{3.11}$$

where $\delta x^0$ is now the increase of the curvature vector at its tip, while the tail is fixed on the evolute. It is evident that the angle of ascent does not change on these straight lines, so that

$$\varepsilon_{|0} = 0 \cdot \tag{3.12}$$

For the anholonomic line element of these straight lines we obtain with (3.11)

$$dx^{0^2} = dx^{0'^2} + dx^{1'^2} = \left[\cot^2\varepsilon + 1\right] dx^{\underline{1}^2} = \frac{1}{\sin^2\varepsilon} a_R^2 dr^2 . \tag{3.13}$$

From this and

$$\begin{aligned} dx^{0'} &= \cot\varepsilon\, a_R dr = \cos\varepsilon\, dx^0 \\ x^{1'} &= r\cos\vartheta \\ x^{2'} &= A\sin\vartheta\cos\varphi \\ x^{3'} &= A\sin\vartheta\sin\varphi \end{aligned} \tag{3.14}$$

we are able to re-derive the components of the anholonomic normal vector. With the help of (3.12) and

$$\partial_0 = \sin\varepsilon \frac{\partial}{a_R \partial r} \tag{3.15}$$

we obtain

$$n^{a'} = x^{a'}{}_{|0} = \left\{\cos\varepsilon,\ \sin\varepsilon\cos\theta,\ \sin\varepsilon\sin\theta\right\} \cdot$$

It is also possible to determine the generalized Frenet formulae for the anholonomic surface. If we use the space-like parts of the connexion coefficients derived in former papers and if we define

$$\frac{\delta m^{a'}}{\delta s} = x^{a'}{}_{||11} = x^{a'}{}_{|11} - Y_{11}{}^{2}\, x^{a'}{}_{|2}$$
$$\frac{\delta n^{a'}}{\delta s} = x^{a'}{}_{||01} = x^{a'}{}_{|01} \quad ,$$
$$\frac{\delta b^{a'}}{\delta s} = x^{a'}{}_{||21} = x^{a'}{}_{|21} - Y_{12}{}^{1}\, x^{a'}{}_{|1}$$

we obtain the generalized Frenet formulae in tetrad representation

$$\begin{aligned} \frac{\delta m^{a'}}{\delta s} &= \kappa n^{a'} \\ \frac{\delta n^{a'}}{\delta s} &= -\kappa m^{a'} + \tau b^{a'} \\ \frac{\delta b^{a'}}{\delta s} &= -\tau n^{a'} \quad . \end{aligned} \qquad (3.16)$$

$$\kappa = -\frac{1}{\rho_S}, \quad \tau = \frac{1}{\rho_H} \sin\varepsilon \cos\varepsilon$$

Using the higher dimensional covariant derivation we find that the simple relation

$$x^{a'}{}_{|||bc} = 0$$

contains the Frenet formulae and also stands for the definition of all higher dimensional connexion coefficients. All these equations can be extended to include more dimensions. Next we will investigate the time-like part of the Kerr metric.

If we envisage the [1',2',4'] – subspace and if we rotate the radial integral line at the semi minor axis about the $x^{1'}$–axis through the angle iψ and if we also rotate the correlated evolute, we get a double surface connected by the curvature vectors. The imaginary circles (hyperbolae of constant curvature) generated by this rotation have the line element

$$dx^4 = \rho_S \cos\varepsilon\, di\psi = i a_S dt \, . \qquad (3.17)$$

If we to take into account Eq. (3.17) we obtain the four-dimensional line element (2.3) and performing the transformation (2.3) we get the Kerr line element. Thus, it is possible to embed the Kerr geometry in a five-dimensional flat space without violating the theorem of Eisenhart and Kasner. This theorem relates to a single surface and demands six dimensions at least for an embedding of a Ricci flat model in a flat space. Our double surface theory [3,4] does with five dimensions but demands two extra variables $x^{0'}, \bar{x}^{0'}$.

# 4. SUMMARY

We have shown that the embedding of the Kerr geometry in a five-dimensional space is possible and we have also shown that the embedded surfaces are in a close relation to the surfaces of the Schwarzschild geometry, discussed in former papers. Previously, we already had derived the Kerr line element and the field equations from a four-dimensional pseudo-hypersphere embedded in a five-dimensional flat space without the knowledge of the structure of the surfaces, derived in the present paper. We had used a projector technique for mapping that hypersphere to elliptically squashed Schwarzschild-like surfaces. It seems to us that this projector technique is a powerful tool for treating gravitational models.

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