ARG Austrian Reports on Gravitation

ARG-2002-01

KERR GEOMETRY I. A FLAT KERR-LIKE MODEL

Rainer Burghardt*

Keywords: Kerr metric, differential rotation law, centrifugal and Coriolis forces.

Contents

We study a flat Kerr-like model for a better understanding of the Kerr geometry. We show that the metric in elliptical co-ordinates provides the mathematical frame for the field of a rotating object. The field strengths are the relativistic generalizations of the centrifugal and Coriolis forces of the classical mechanics. But new forces also appear as the consequence of the differential rotation law. All these forces are governed by Maxwell-like field equations.

* e-mail: arg@arg.at, home page: http://arg.or.at/

1. INTRODUCTION

The Schwarzschild metric and the Kerr metric are the most important ones for describing the nature of gravitation. While the Schwarzschild metric is fairly well understood, the Kerr metric still needs some clarification. So we start with a simplified model. Setting the mass parameter of the Kerr metric to zero, we get a flat model which shows a lot of features of the Kerr geometry: centrifugal and Coriolis fields and new fields, having their origins in the differential rotation law. This model has the great advantage that the velocity of a co-rotating observer vanishes at infinity so that his velocity never exceeds the velocity of light and does not violate the principles of relativity. In the last decades many searchers [1,2,3] tried to adapt a special rotation law to avoid superluminous effects. The Kerr metric provides such a law in a natural way.

2. FLAT METRIC AND ROTATION

The Kerr metric in Boyer-Lindquist co-ordinates can be reduced to a flat space metric by setting the mass parameter M to zero:

$$ds^2 = dx^{1^2} + dx^{2^2} + \left[\alpha dx^3 + i\alpha\omega\sigma dx^4\right]^2 + \left[-i\alpha\omega\sigma dx^3 + \alpha dx^4\right]^2, \quad (2.1)$$

where

$$dx^1 = \frac{\Lambda}{A}dr, \quad dx^2 = \Lambda d\vartheta, \quad dx^3 = \sigma d\varphi, \quad dx^4 = idt \quad (2.2)$$

$$\alpha = \frac{A}{\Lambda}, \quad \omega = \frac{a}{A^2}, \quad \sigma = A\sin\vartheta, \quad (2.3)$$

A and r being the semi-axes of confocal ellipses with eccentricity a [4]

$$A^2 = r^2 + a^2, \quad \Lambda^2 = r^2 + a^2\cos^2\vartheta. \quad (2.4)$$

By use of the rectilinear co-ordinates $x^{i'}$

$$\begin{aligned} x^{3'} &= a\,\mathrm{ch}\eta\sin\vartheta\cos\varphi = A\sin\vartheta\cos\varphi \\ x^{2'} &= a\,\mathrm{ch}\eta\sin\vartheta\sin\varphi = A\sin\vartheta\sin\varphi \\ x^{1'} &= a\,\mathrm{sh}\eta\cos\vartheta = r\cos\vartheta \\ r &= a\,\mathrm{sh}\eta, \; A = a\,\mathrm{ch}\eta \end{aligned} \quad (2.5)$$

the 3-dimensional space is parametrized by the confocal ellipsoids and hyperboloids of revolution (the $[x^{1'}, x^{2'}]$ plane is drawn in Fig. 1).

$$\frac{(x^{2'})^2+(x^{3'})^2}{a^2 ch^2\eta}+\frac{(x^{1'})^2}{a^2 sh^2\eta}=1, \qquad \frac{(x^{2'})^2+(x^{3'})^2}{a^2\sin^2\vartheta}-\frac{(x^{1'})^2}{a^2\cos^2\vartheta}=1. \tag{2.6}$$

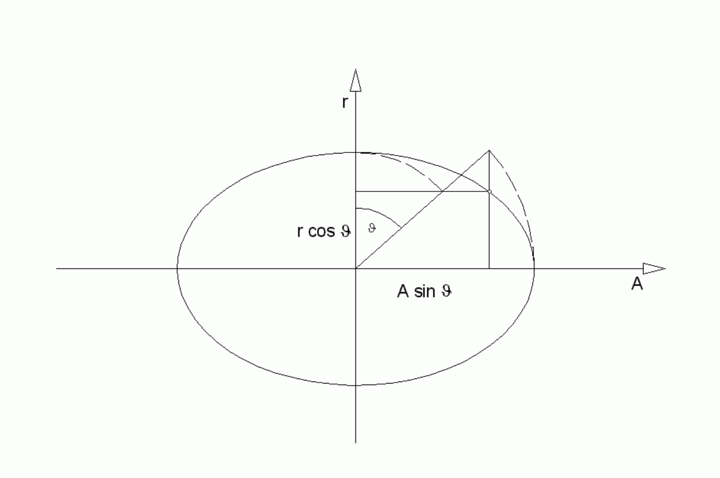

Fig. 1

and the line element reads as

$$(dx^{1'})^2+(dx^{2'})^2+(dx^{3'})^2+(dx^{4'})^2=a^2\left(sh^2\eta+\cos^2\vartheta\right)\left(d\eta^2+d\vartheta^2\right)+a^2 ch^2\eta\sin^2\vartheta d\varphi^2-dt^2 \tag{2.7}$$

or

$$ds^2=\frac{\Lambda^2}{A^2}dr^2+\Lambda^2 d\vartheta^2+A^2\sin^2\vartheta d\varphi^2-dt^2, \tag{2.8}$$

which also can be derived from (2.1) by performing a Lorentz transformation with (2.3). ω is the angular velocity of the rotating frame, α the correlated Lorentz factor and σ the radius of the observer's circular orbit. We find the factor of the radial[1] part of the line element (2.8)

$$a_R^2 = \frac{\Lambda^2}{A^2}=1-\frac{a^2}{A^2}\sin^2\vartheta=1-\omega^2\sigma^2 \tag{2.9}$$

to carry most of the information we need for the implementation of the rotation. Relation (2.9) leads us to suitable quantities (2.3) and we will show in the following that the rotating model (2.1) exhibits a lot of features of the Kerr metric.

As the vectors of curvature of the ellipses and hyperbolae are

$$\rho_E=\frac{\Lambda^3}{rA}, \quad \rho_H=-\frac{\Lambda^3}{a^2\sin\vartheta\cos\vartheta}, \tag{2.10}$$

[1] By 'radial' we understand the direction tangential to the hyperbolae.

we find simple expressions for the connexion coefficients in tetrad representation

$$
\begin{gathered}
A_{mn}{}^{s} = B_{mn}{}^{s} + N_{mn}{}^{s} + C_{mn}{}^{s} \\
B_{mn}{}^{s} = b_m B_n b^s - b_m b_n B^s,\; N_{mn}{}^{s} = m_m N_n m^s - m_m m_n N^s,\; C_{mn}{}^{s} = c_m C_n c^s - c_m c_n C^s \\
B_1 = \frac{1}{\Lambda}\Lambda_{|1} = \frac{1}{\rho_E},\; N_2 = \frac{1}{\Lambda}\Lambda_{|2} = \frac{1}{\rho_H},\; C_1 = \frac{1}{\sigma}\sigma_1 = \frac{r}{A\Lambda},\; C_2 = \frac{1}{\sigma}\sigma_2 = \frac{1}{\Lambda}\cot\vartheta \quad , \quad (2.11) \\
\sigma_1 = \sigma_{|1} = \frac{r}{\Lambda}\sin\vartheta, \quad \sigma_2 = \sigma_{|2} = \frac{A}{\Lambda}\cos\vartheta
\end{gathered}
$$

where m_n= {1,0,0,0} , b_n= {0,1,0,0}, c_n= {0,0,1,0} are the orthogonal unit vectors in the three space-like directions, tangential to the hyperbolae, ellipses and circles with the radius σ. From the flat Ricci tensor

$$R_{mn} \equiv 0$$

we get

$$
\begin{gathered}
\left[N_{n|m} - m_m N_{n|s} m^s + N_n N_m\right] + \left[B_{n|m} - b_m B_{n|s} b^s + B_n B_m\right] \\
+ m_n m_n \left[N^s{}_{|s} + N^s N_s\right] + b_n b_n \left[B^s{}_{|s} + B^s B_s\right] = 0 \quad , \qquad (2.12) \\
C_{n\underset{3}{||}m} + C_n C_m = 0
\end{gathered}
$$

and by contraction the equations for the curvature of these three curves mentioned above

$$\left[N^s{}_{\underset{2}{||}s} + N^s N_s\right] + \left[B^s{}_{\underset{2}{||}s} + B^s B_s\right] = 0, \quad \left[C^s{}_{\underset{3}{||}s} + C^s C_s\right] = 0\,. \qquad (2.13)$$

The graded derivatives [5] are defined by

$$m_{m\underset{1}{||}n} = m_{m|n}, \quad b_{m\underset{2}{||}n} = b_{m|n}, \quad c_{m\underset{3}{||}n} = c_{m|n} - N_{nm}{}^{s} c_s - B_{nm}{}^{s} c_s \qquad (2.14)$$

and have the properties

$$m_{m\underset{1}{||}n} = 0,\; b_{m\underset{2}{||}n} = 0,\; c_{m\underset{3}{||}n} = 0\,. \qquad (2.15)$$

To verify (2.12) more investigations on the geometry have to be done.

3. ROTATIONAL QUANTITIES IN THE STATIC MODEL

As the space-like part of the model is governed by the vectors of curvature, we study these quantities in more detail. From (2.10) we get

$$\rho_{E|1} = 1 - \rho_E{}^2\,\Omega^{m3}\Omega_{3m}, \quad \rho_{E|2} = 3\frac{\rho_E}{\rho_H}, \quad \rho_{H|1} = 3\frac{\rho_H}{\rho_E}, \quad \rho_{H|2} = 1 + \rho_H{}^2\,\Omega^{m3}\Omega_{3m}, \tag{3.1}$$

where

$$\Omega_{nm} = -H_{mn} - D_{mn}, \; H_{mn} = 2i\alpha^2\omega\sigma_{[m}c_{n]}, \; D_{mn} = i\alpha^2\omega_{|m}\sigma c_n, \; \omega_{|1}\sigma = -2\omega\sigma_1, \; \omega_{|2}\sigma = 0. \tag{3.2}$$

H is an antisymmetric quantity, the relativistic generalization of the Coriolis field, while D is asymmetric and results from the differential rotation law. We find

$$N_{2|2} + N_2N_2 = -\Omega^{m3}\Omega_{3m}, \; B_{1|1} + B_1B_1 = \Omega^{m3}\Omega_{3m}, \tag{3.3}$$

which explains the first equation (2.12). In the next Section we will show that the quantity Ω arises from the rotating metric (2.1), but at the moment we are searching for a deeper understanding for (3.1). We start with the spherical parameterization of the ellipsoids by

$$\begin{aligned} X^{3'} &= \rho_E \sin\theta\cos\varphi \\ X^{2'} &= \rho_E \sin\theta\sin\varphi, \qquad\qquad X_{m'}X^{m'} = \rho_E{}^2, \\ X^{1'} &= \rho_E\cos\theta \end{aligned} \tag{3.4}$$

the surfaces $\rho_E = \text{const.}$ being spheres with the radius vector $X^{m'}$. In spherical co-ordinates the metric reads as follows

$$\begin{aligned} ds^2 &= d\rho_E{}^2 + \rho_E{}^2 d\theta^2 + \rho_E{}^2\sin^2\theta\, d\varphi^2 \\ dX^1 &= d\rho_E, \quad dX^2 = \rho_E d\theta, \quad dX^3 = \rho_E\sin\theta\, d\varphi, \\ \Phi_{,m} &= \hat{\partial}_m\Phi = \frac{\partial\Phi}{\partial X^m} = \left\{\frac{\partial}{\partial\rho_E}, \frac{\partial}{\rho_E\partial\theta}, \frac{\partial}{\rho_E\sin\theta\partial\varphi}\right\} \end{aligned} \tag{3.5}$$

θ being the off-axis angle of the radius ρ_E. The connexion coefficients read as

$$\begin{aligned} \hat{A}_{mn}{}^s &= \hat{B}_{mn}{}^s + \hat{C}_{mn}{}^s \\ \hat{B}_{mn}{}^s &= b_m\hat{B}_n b^s - b_m b_n\hat{B}^s, \quad \hat{C}_{mn}{}^s = c_m\hat{C}_n c^s - c_m c_n\hat{C}^s \\ \hat{B}_n &= \left\{\frac{1}{\rho_E}, 0, 0\right\}, \quad \hat{C}_n = \left\{\frac{1}{\rho_E}, \frac{1}{\rho_E}\cot\theta, 0\right\} \end{aligned} \tag{3.6}$$

and the field equations read as

$$\hat{B}_{\underset{2}{n;m}}+\hat{B}_n\hat{B}_m=0,\quad \hat{C}_{\underset{3}{n;m}}+\hat{C}_n\hat{C}_m=0,\quad \hat{B}_{\underset{2}{n;m}}=\hat{B}_{n,m},\ \hat{C}_{\underset{3}{n;m}}=\hat{C}_{n,m}-\hat{B}_{mn}{}^{s}\hat{C}_s. \tag{3.7}$$

We retrieve the elliptic system by differentiating (2.10) and demanding that now ρ_E and θ are function of r and ϑ:

$$\rho_E=\rho_E(r,\vartheta),\ \theta=\theta(r,\vartheta). \tag{3.8}$$

In this way obtain

$$\sigma_1=\frac{r}{\Lambda}\sin\vartheta=\sin\theta,\ \sigma_2=\frac{A}{\Lambda}\cos\vartheta=\cos\theta \tag{3.9}$$

$$d\rho_E=\left(1-\rho_E{}^2\,\Omega^{m3}\Omega_{3m}\right)dx^1+3\frac{\rho_E}{\rho_H}dx^2,\ d\theta=-\frac{1}{\rho_H}dx^1+\frac{1}{\rho_E}dx^2. \tag{3.10}$$

The equations of the evolutes of the ellipses are

$$\left(r\,\overline{x}^{1'}\right)^{\frac{2}{3}}+\left(A\overline{x}^{2'}\right)^{\frac{2}{3}}=\left(a^2\right)^{\frac{2}{3}}, \tag{3.11}$$

in agreement with

$$\overline{x}^{1'}=-\frac{a^2}{r}\cos^3\vartheta,\ \overline{x}^{2'}=\frac{a^2}{A}\sin^3\vartheta. \tag{3.12}$$

By the differentiation of these relations we find, with the help of (3.9),

$$\begin{aligned} d\overline{x}^{1'}&=\frac{a^2}{r^2}\cos^2\vartheta\cos\theta dx^1+3\frac{a^2}{r^2}\cos^2\vartheta\sin\theta dx^2\\ d\overline{x}^{2'}&=-\frac{a^2}{A^2}\sin^2\vartheta\sin\theta dx^1+3\frac{a^2}{A^2}\sin^2\vartheta\cos\theta dx^2\end{aligned}. \tag{3.13}$$

Rotating locally through θ we arrive at

$$d\overline{x}^1=\rho_E{}^2\,\Omega^{m3}\Omega_{3m}dx^1-3\frac{\rho_E}{\rho_H}dx^2,\quad d\overline{x}^2=\frac{\rho_E}{\rho_H}dx^1. \tag{3.14}$$

The vector X of (3.4) may be described by its tip and tail $X^m=x^m-\overline{x}^m$. Its differentials in elliptical co-ordinates are

$$dX^1 = d\rho_E = dx^1 - d\bar{x}^1 = \left(1 - \rho_E{}^2\Omega^{m3}\Omega_{3m}\right)a_R dr + 3\frac{\rho_E}{\rho_H}\Lambda d\vartheta$$
$$dX^2 = \rho_E d\theta = dx^2 - d\bar{x}^2 = \Lambda d\vartheta - \frac{\rho_E}{\rho_H}a_R dr \qquad . \qquad (3.15)$$

We remark that the rotational object Ω_{3m} is the contribution of the evolutes of the ellipsoids. Considering also

$$dX^3 = \hat{\sigma}d\varphi = dx^3 - d\bar{x}^3, \quad \hat{\sigma} = \rho_E \sin\theta = a_R^2\,\sigma = \sigma - \bar{\sigma}, \quad \bar{\sigma} = \omega^2\sigma^2 \cdot \sigma = \frac{a^2}{A}\sin^3\vartheta\,, \quad (3.16)$$

we are able to deduce the whole geometry of the ellipsoids of revolution from a spherical geometry by introducing the projectors

$$dX^m = \mathfrak{p}^m_n dx^n, \quad \partial_n = \mathfrak{p}^m_n \hat{\partial}_m$$
$$\mathfrak{p}^1_1 = 1 - \rho_E{}^2\Omega^{m3}\Omega_{3m}, \quad \mathfrak{p}^1_2 = 3\frac{\rho_E}{\rho_H}, \quad \mathfrak{p}^2_1 = -\frac{\rho_E}{\rho_H}, \; \mathfrak{p}^2_2 = 1, \quad \mathfrak{p}^3_3 = a_R^2\;, \qquad (3.17)$$

which leads us to a very simple formulation of the equations (3.10), (3.15). The projectors are acting on all indices correlated to differentials or derivatives. One easily finds the relation of the spherical and elliptical connexion coefficients by

$$A_{mn}{}^s = \mathfrak{p}^r_m \hat{A}_{rn}{}^s$$
$$B_1 = A_{21}{}^2 = \mathfrak{p}^2_2\hat{A}_{21}{}^2 = \frac{1}{\rho_E}, \quad N_2 = A_{12}{}^1 = \mathfrak{p}^2_1\hat{A}_{22}{}^1 = \frac{1}{\rho_H} \qquad , \quad (3.18)$$
$$C_1 = A_{31}{}^3 = \mathfrak{p}^3_3\hat{A}_{31}{}^3 = \frac{1}{\rho_E}a_R^2 = \frac{r}{\Lambda A}, \quad C_2 = A_{32}{}^3 = \mathfrak{p}^3_3\hat{A}_{32}{}^3 = \frac{1}{\rho_E}a_R^2\cot\theta = \frac{1}{\Lambda}\cot\vartheta$$

or also

$$B_{1\|1} = \mathfrak{p}^1_1\left(\mathfrak{p}^2_1\hat{B}_1\right)_{,1} = -B_1B_1 + \Omega^{m3}\Omega_{3m}\,.$$

The tensors of curvature are related by

$$\mathfrak{p}^p_r\mathfrak{p}^q_m\,\hat{R}_{pqn}{}^s = R_{rmn}{}^s + 2\hat{A}_{pn}{}^s\,\mathfrak{p}^{\;p}_{[r\|m]} \equiv 0\,. \qquad (3.19)$$

All components of the last term vanish. It is of some interest that two new quantities appear in differentiating $\mathfrak{p}^3_3$. From the radial coefficient of the metric (2.9) we derive

$$\frac{1}{\alpha_R}\alpha_{R|m} = F_m + D_m, \quad F_m = \alpha_R{}^2\,\omega^2\sigma\sigma_m, \quad D_m = \alpha_R{}^2\,\omega\,\omega_{|m}\sigma^2, \qquad (3.20)$$

where F is the centrifugal field normal to the $x^{3'}$-axis, which will be the rotation axis of the rotating system. D results from the differential rotation law and is normal to the elliptical surfaces. We have shown that the geometry of the static model provides all quantities we need to describe the physics of the rotating system. In analyzing the Kerr metric we find the same quantities (3.2), (3.10) and we will make use of the projection technique developed in this Section.

4. THE ROTATING SYSTEM

Performing a Lorentz transformation with parameters (2.3) the connexion coefficients (2.11) are endowed with new components

$$\begin{gathered} A_{mn}{}^{s} = B_{mn}{}^{s} + N_{mn}{}^{s} + C_{mn}{}^{s} + F_{mn}{}^{s} + H_{mn}{}^{s} + E_{mn}{}^{s} \\ F_{mn}{}^{s} = c_m F_n c^s - c_m c_n F^s, \quad E_{mn}{}^{s} = -\left[u_m F_n u^s - u_m u_n F^s\right] \\ H_{mn}{}^{s} = H_{mn} u^s + H^s{}_m u_n + H^s{}_n u_m + D_{mn} u^s - D_m{}^s u_n \end{gathered} \tag{4.1}$$

where u_m is the unit vector of the local time direction and also the 4-velocity of the co-rotating observer. As the rotation is only attached to the geometry and appears dynamically in the 'Einstein field equations' of the flat space, the new field equations for the rotation should decouple:

$$\begin{gathered} F^m{}_{\underset{4}{||}m} - F^m F_m - \Omega^{mn}\Omega_{nm} = 0 \\ \Omega^{mn}{}_{\underset{4}{||}m} + 2\Omega^{[nm]} F_m = 0 \end{gathered}, \tag{4.2}$$

which is the first set of Maxwell's equations of rotation. The second set is

$$\begin{gathered} F_{[m\underset{4}{||}n]} + D_{[m\underset{4}{||}n]} = 0, \quad F_{[m\underset{4}{||}n]} + 2\Omega_{3[m}\Omega_{n]3} = 0 \\ \Omega_{[mn\underset{4}{||}s]} + \Omega_{[mn}F_{s]} = 0 \end{gathered}, \tag{4.3}$$

where the graded derivative is

$$F_{m\underset{4}{||}n} = F_{m|n} - \left(N_{nm}{}^{s} + B_{nm}{}^{s} + C_{nm}{}^{s} + F_{nm}{}^{s}\right)F_s .$$

The quadratic terms in (4.2) are the field energy and the Poynting vector of the rotating system. The field energy produces a negative field mass in the center of rotation, which is repulsive and the source of the centrifugal field. The relative circulation of the universe also produces the Coriolis field, which has a high negative mass density. These energy densities occur only in the rotating frames, thus the localizability of the field energy depends to a high degree on the choice of the reference frame. These effects result from the non-linearity of the field equations and were investigated by F. Hund [6], and H. Hönl and H. Dehnen [7]. Field energy and Poynting vector satisfy conservation laws

$$\frac{\partial}{\partial t}\left(F^m F_m + \Omega^{mn}\Omega_{nm}\right) = 0, \; \left(2\Omega^{[nm]}F_m\right)_{\underset{4}{\|n}} = 0 . \tag{4.4}$$

We will work out for the Kerr geometry a fairly similar structure.

5. CONCLUSIONS

A methodical examination of the flat elliptical geometry shows that the basic fields are the curvatures of the ellipses ($1/\rho_E$), hyperbolae ($1/\rho_H$) and circles ($1/\sigma$). They obey simple covariant field equations, which could be derived from the flat Ricci tensor. Moreover, one has also to consider the contributions of the evolutes of the ellipses. They explain the occurrence of the rotational quantities in the static case. This is a first step towards an understanding of the Kerr geometry. In a further paper, we will make use of these results.

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