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INTERIOR SCHWARZSCHILD SOLUTION AND FREE FALL

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Contents

The velocity of a freely falling observer falling in a tube through the center of a stellar object described by the Schwarzschild interior solution is calculated. In a comoving system one can investigate the tidal forces that act on the freely falling observer.

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1. INTRODUCTION

In the following, we want to consider a motion in the interior of a stellar object which formally corresponds to the free fall in the exterior. Without a doubt, this will be a gedankenexperiment because inside the matter no free motion is possible. Thus, we imagine a tube bored through the center of the stellar object through which we let fall a test particle from arbitrary positions from the exterior. Since we do not admit forces further than gravitational forces, the test particle will accelerate to the center of the stellar object, will come out at the opposite side of the object, and will come to rest in a position symmetrically to the starting point. Its motion will reverse, the test particle will move back and forth. Relevant computations have been performed in the context of Newton's theory. Similar considerations that concern the interior Schwarzschild field are not known to us. We try to find an ansatz for the speed of a test particle inside the matter.

2. FREE FALL

Firstly, we turn to the simpler problem, i.e. that a test particle starts freely falling in from infinity. Its speed on the surface of the stellar object is[1]

$$v_g = -\sqrt{\frac{2M}{r_g}} \tag{2.1}$$

and must coincide with the initial speed on the surface of the interior. In a previous paper [1] we have shown that freely falling observers cannot experience the force of gravity. This force is nullified by a force having its origin in the acceleration of the observer. We obtain such a force G from the Lorentz transformation which connects the system of a static observer with that of a freely falling observer. For the latter G - E = 0 must be valid, whereby E means the force of gravity inside the stellar object. Thus, we already found a way to determine the velocity of an observer freely falling through the tube. From the exterior Schwarzschild solution we know that the speed of a freely falling observer is defined by the geometry. The redshift factor is the reciprocal of the Lorentz factor of the motion of the freely falling observer and has for the interior solution the form

$$a_T = \frac{1}{2}\left[3\cos\eta_g - \cos\eta\right] \ . \tag{2.2}$$

The space-like part of the interior solution is geometrically described by a cap of a sphere with the aperture angle η_g. $\eta < \eta_g$ is an arbitrary polar angle fixing the position of an observer. More details on this subject can be found in our paper [2]. If one wants to

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[1] The marker g denotes the value of a quantity on the boundary surface of the interior and exterior solutions.

replace in (2.2) the trigonometric functions by the standard Schwarzschild co-ordinate r, first one has to use the relation

$$\cos^2\eta = 1 - \frac{r^2}{\mathcal{R}^2} \tag{2.3}$$

and then to continue with the values at the boundary surface

$$\rho_g = 2\mathcal{R} = \sqrt{\frac{2r_g^3}{M}}, \quad \mathcal{R} = \sqrt{\frac{r_g^3}{2M}} \ . \tag{2.4}$$

Finally, one has for the Lorentz factor

$$\alpha_T(r) = \frac{2}{3\sqrt{1-\frac{2M}{r_g}} - \sqrt{1-\frac{2Mr^2}{r_g^3}}} \tag{2.5}$$

and for the speed of the falling observer

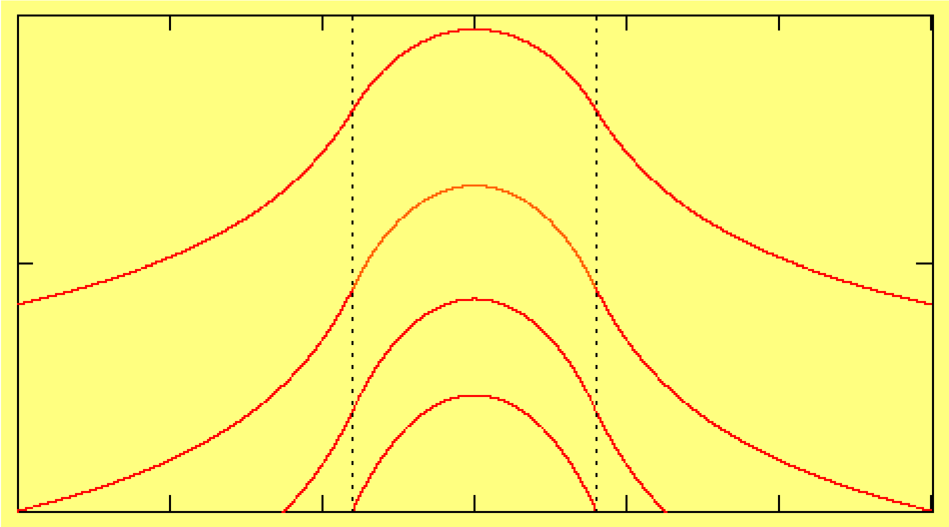
$$v_T(r) = -\sqrt{1 - \frac{1}{4}\left(3\sqrt{1-\frac{2M}{r_g}} - \sqrt{1-\frac{2Mr^2}{r_g^3}}\right)^2} \ . \tag{2.6}$$

One sees immediately that this relation coincides with (2.1) for $r = r_g$. In paper [1] we have set up a formula for a freely falling object which does not come from infinity, but started its motion from an arbitrary position r_0 within the exterior Schwarzschild region. This formula simply has to be transferred for the case where the test particle continues its motion into the inside. One has

$$v_I(r, r_0) = \frac{v_T - \left(-\sqrt{\frac{2M}{r_0}}\right)}{1 - v_T\left(-\sqrt{\frac{2M}{r_0}}\right)} \tag{2.7}$$

and one finds the relation (2.6) again for $r_0 = \infty$. If the body starts on the surface of the stellar object one has $v_I(v_g, v_g) = 0$.

We show some examples in the Figure below. The surface of the stellar object is indicated by the dashed lines.



In order to bring in further considerations on the freely falling observer, we limit ourselves to the simpler case where the test particle starts its fall from the infinite, but we study only the interior region of the stellar object. For this purpose we consult the formula (2.2) and

$$v = -\sqrt{1-\frac{1}{4}\left(3\cos\eta_g - \cos\eta\right)^2}, \quad \alpha = a_T^{-1} \tag{2.8}$$

whereby we omit the marker $_T$ from now on. Thus, we already have made accessible the parameters of a Lorentz transformation

$$L^{1}_{1'} = \alpha, \quad L^{1}_{4'} = -i\alpha v, \quad L^{4}_{1'} = i\alpha v, \quad L^{4}_{4'} = \alpha \tag{2.9}$$

which demonstrate the connection between the static and the falling observers. The unit vectors of the static system in the 1- and 4-directions

$$m_m = \{1,0,0,0\}, \quad u_m = \{0,0,0,1\} \tag{2.10}$$

take from the view of the falling observer the form

$$m_{m'} = \{\alpha,0,0,-i\alpha v\}, \quad u_{m'} = \{i\alpha v,0,0,\alpha\}\,. \tag{2.11}$$

The freely falling observer has for his own system

$$'m_{m'} = \{1,0,0,0\}, \quad 'u_{m'} = \{0,0,0,1\}, \tag{2.12}$$

however, his unit vectors are measured by the static observer as

$$'m_m = \{\alpha,0,0,i\alpha v\}, \quad 'u_m = \{-i\alpha v,0,0,\alpha\}\,. \tag{2.13}$$

The last four formulae contain the basic laws of the relativity theory.

To get more information on the forces of the falling system we start with the static metric of the interior Schwarzschild solution

$$ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \sin^2\eta\, d\vartheta^2 + \mathcal{R}^2 \sin^2\eta \sin^2\vartheta\, d\varphi^2 + a_T^2 dit^2 \ . \tag{2.14}$$

We read the bein vectors from the metric and compute the field strengths of the static system which are transformed into the falling system by (2.9). For the partial derivatives one has

$$\partial_{1'} = \alpha \frac{\partial}{\mathcal{R}\partial\eta}, \quad \partial_{4'} = -i\alpha v \frac{\partial}{\mathcal{R}\partial\eta} \ . \tag{2.15}$$

The Ricci-rotation coefficients, which describe the curvatures of the surface maintain their geometrical properties under Lorentz transformations. Thus, they transform like tensors. If one subjects the covariant derivatives to a Lorentz transformation the Lorentz term

$$\Phi_{m'\|n'} = L^{m\ n}_{m'n'}\Phi_{m\|n} = \left[\Phi_{m'|n'} - L^{s'}_{s} L^{s}_{m'|n'}\Phi_{s'}\right] - A_{n'm'}{}^{s'}\Phi_{s'}\,, \quad A_{n'm'}{}^{s'} = L^{n\ m\ s'}_{n'm's} A_{nm}{}^{s}\ , \tag{2.16}$$

arises that has its origin in the not-constant parameters of the Lorentz transformation. Thus, the basics of the free fall through the interior of a stellar object are outlined in short. For the further treatment of the problem we can hark back to the results of the free fall in the exterior field. However, we have to point out the differences.

Having computed the field quantities

$$\begin{aligned} B_{m'} &= \left\{\alpha \frac{1}{\mathcal{R}}\cot\eta, 0, 0, -i\alpha v \frac{1}{\mathcal{R}}\cot\eta\right\} \\ C_{m'} &= \left\{\alpha \frac{1}{\mathcal{R}}\cot\eta, \frac{1}{\mathcal{R}\sin\eta}\cot\vartheta, 0, -i\alpha v \frac{1}{\mathcal{R}}\cot\eta\right\} \\ E_{m'} &= \left\{-\alpha \frac{1}{\rho_g}\frac{\sin\eta}{a_T}, 0, 0, i\alpha v \frac{1}{\rho_g}\frac{\sin\eta}{a_T}\right\} \end{aligned} \tag{2.17}$$

from the static system with the help of the Lorentz transformation we only have to deal with the Lorentz term

$$L_{n'm'}{}^{s'} = L^{s'}_{s} L^{s}_{m'|n'} \ . \tag{2.18}$$

It has the components

$$L_{4'1'}{}^{4'} = G_{1'}, \quad L_{1'4'}{}^{1'} = -i\frac{1}{v}G_{1'}, \quad G_{1'} = -\alpha \frac{1}{\rho_g}\frac{\sin\eta}{a_T} \ . \tag{2.19}$$

Since it is evident from (2.16) that the Lorentz term joins the connexion coefficients one obtains

$$L_{4'1'}{}^{4'} + A_{4'1'}{}^{4'} = G_{1'} - E_{1'} = 0 \ . \tag{2.20}$$

The two quantities E and G have different sources, G a kinematic one and an E a geometrical one. They are formally identical and nullify each other. Inside a stellar object a freely falling observer cannot experience the force of gravity, just as can a freely falling observer in the exterior field. Thus, we have fulfilled a requirement proposed at the beginning.

Since the Ricci tensor and thus the Einstein tensor are Lorentz invariant, we obtain well-known structures for the field equations [1] which we do not need to repeat here. Thus, the stress-energy tensor has the form

$$T_{m'n'} = -m_{n'}m_{m'}p - b_{n'}b_{m'}p - c_{n'}c_{m'}p + u_{n'}u_{m'}\mu_0 \tag{2.21}$$

whereby the values (2.11) are to be considered. Written in more detail

$$T_{m'n'} = \begin{pmatrix} -p & & & \\ & -p & & \\ & & -p & \\ & & & \mu_0 \end{pmatrix} + \begin{pmatrix} -\alpha^2 v^2 & & & i\alpha^2 v \\ & 0 & & \\ & & 0 & \\ -i\alpha^2 v & & & \alpha^2 v^2 \end{pmatrix} (p + \mu_0) \tag{2.22}$$

one recognizes that the stress-energy tensor splits into a static and into a kinematic part. p is the hydrostatic pressure of the fluid sphere, which is taken in rough approximation for the stellar object, and μ_0 is its energy density. $\mu_0 + p$ is the total energy density which consists of the hydrostatic pressure-energy density and the energy density attributed to the mass of the stellar object.

After some algebra one obtains with the help of the expressions (2.2), (2.8), and with the relation $E_{m'} = \alpha^{-1}\alpha_{|m'}$ which has been treated in [1] the conservation law

$$T_{m'}{}^{n'}{}_{||n'} = -p_{|m'} + (p + \mu_0) E_{m'} = 0 \ . \tag{2.23}$$

It corresponds to the conservation law of the static system

$$p_{|\alpha} = (p + \mu_0) E_\alpha, \quad \dot{p} = 0, \quad \dot{\mu} = 0, \quad \alpha = 1,2,3 \ . \tag{2.24}$$

3. TIDAL FORCES

Up to now we have strictly followed the investigations of the static system by observing the components of the static field strengths in the falling system. As the first new result we have found that the freely falling observer experiences no force of gravity. Similarly to the theory of the exterior field, we will re-write the formulae in such a way that we obtain relations for the tidal forces which affect the freely falling observer in place of the force of gravity. We obtain the first component of the tidal forces from the interplay of the force of gravity and of the Lorentz term.

The relation (2.20) can be formulated more generally

$$\begin{aligned} G_{n'm'}{}^{s'} &= u_{n'}G_{m'}u^{s'} - u_{n'}u_{m'}G^{s'} = {}'u_{n'}G_{m'}\,{}'u^{s'} - {}'u_{n'}\,{}'u_{m'}G^{s'} \\ E_{n'm'}{}^{s'} &= -u_{n'}E_{m'}u^{s'} + u_{n'}u_{m'}E^{s'} = -{}'u_{n'}E_{m'}\,{}'u^{s'} + {}'u_{n'}\,{}'u_{m'}E^{s'} \end{aligned} \ . \tag{3.1}$$

One has

$$L_{n'm'}{}^{s'} = Q_{n'm'}{}^{s'} + G_{n'm'}{}^{s'} \ . \tag{3.2}$$

Since

$$G_{n'm'}{}^{s'} + E_{n'm'}{}^{s'} = 0 \tag{3.3}$$

one has in the Ricci-rotation coefficients

$$L_{n'm'}{}^{s'} + E_{n'm'}{}^{s'} = Q_{n'm'}{}^{s'} \tag{3.4}$$

wherein $Q_{n'm'}{}^{s'}$ possesses only the one component

$$Q_{1'4'}{}^{1'} = Q_{4'} = \frac{i}{\rho_g}\frac{\sin\eta}{v} \tag{3.5}$$

in accordance with (2.19). On the boundary surface one has

$$Q^{g}_{4'} = \frac{i}{\rho_g}\frac{\sin\eta_g}{v_g} = -\frac{i}{\rho_g} \ . \tag{3.6}$$

This quantity coincides with a component of the second fundamental forms of the surface theory of the exterior solution which describes the shrinking surface that accompanies a freely falling observer. Since we are concerned now and through out only with the freely falling system we omit the primes at the indices and the kernels. The equation (3.5) we write more briefly as $Q_{11} = Q_4$ and we supplement

$$Q_{11} = Q_4, \quad Q_{22} = B_4, \quad Q_{33} = C_4, \quad Q_{[mn]} = 0 \ . \tag{3.7}$$

Thus, we have inferred the complete set of the second fundamental forms of the shrinking surface and at the same time the tidal forces. We split the Ricci-rotation coefficients into

$$A_{mn}{}^{s} = {}^*A_{mn}{}^{s} + Q_m{}^{s}u_n - Q_{mn}u^{s}, \quad A_n = {}^*A_n + u_nQ_s{}^{s} \tag{3.8}$$

and thereby we decompose the Ricci into a purely spatial part and a part which describes the field mechanism of the tidal forces

$$\begin{aligned} R_{mn} = {}^*R_{mn} & \\ & -\left[Q_{mn|s}u^{s} + Q_{mn}Q_s{}^{s}\right] \\ & -u_n\left[Q_s{}^{s}{}_{\hat{\ }\underline{m}} - Q_m{}^{s}{}_{\hat{\ }s}\right] \\ & -u_m\left[{}^*A_{n|s}u^{s} + {}^*A_{sn}{}^{r}Q_r{}^{s}\right] \\ & -u_mu_n\left[Q_s{}^{s}{}_{|m}u^{m} + Q_{rs}Q^{rs}\right] \end{aligned} \ . \tag{3.9}$$

The underlined indices indicate the spatial components of a quantity and the hat denotes the associated 3-dimensional covariant derivative. *R is the 3-dimensional Ricci. Its structure corresponds to the 4-dimensional Ricci. It consists only of 3-dimensional quantities with the appropriate graded derivatives

$$
\begin{aligned}
{}^*R_{mn} = &-\left[{}^*B_{n\underset{2}{\wedge}\underline{m}} + {}^*B_n{}^*B_m\right] - \left[{}^*C_{n\underset{3}{\wedge}\underline{m}} + {}^*C_n{}^*C_m\right] \\
&-b_m b_n\left[{}^*B^s{}_{\underset{2}{\wedge}s} + {}^*B^s{}^*B_s\right] - c_m c_n\left[{}^*C^s{}_{\underset{3}{\wedge}s} + {}^*C^s{}^*C_s\right]
\end{aligned}
\quad . \tag{3.10}
$$

However, the further treatment of the Einstein field equations are more difficult than that of the exterior field. The squared brackets in (3.9) which describe the field mechanism of the tidal forces decouple only partly from the field equations. Thus, one has to compute the Einstein tensor and from it the stress-energy tensor. After having solved the Q-equations one arrives at the expression already computed (2.22). From (3.9) one can isolate a fairly Maxwell-like Q-relation

$$
2Q_{[s}{}^{s}{}_{\wedge \underline{m}]} = \kappa T_{\underline{m}4} \ . \tag{3.11}
$$

On the right side of the equation is the energy-current density of the matter which is coupled to the tidal forces. The geometry does not appear flat inside the matter for a freely falling observer. That was the case for the exterior field because the curvature factor of the metric and the Lorentz factor neutralize each other for the space-like field quantities. This is not the case for the interior field.

4. SUMMARY

We have generalized Newton’s free fall through a stellar object for the case of general relativity by applying the interior Schwarzschild solution. In addition, we have calculated the tidal forces acting on a freely falling observer and we have established the field equations for these tidal forces.

5. REFERENCES

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