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HYBRID COSMOLOGICAL MODELS

Rainer Burghardt*

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Abstract: We show that some of the FRW models are hybrid models. In the hybrid case, spatial parts of the line elements occur, which are typical for non-comoving systems, but the time in these models is the universal cosmological time – a comoving coordinate of a freely falling system. We investigate the line elements of models in the Florides metric and models in the Lemaître metric.

1. INTRODUCTION

Many papers on cosmological models start with the line element

$$ds^2 = \mathcal{R}^2\left(\frac{1}{1-kr^2}dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2\right) - dt^2. \qquad (1.1)$$

Here, $\mathcal{R}$ either is a constant or a time-dependent variable and $k = \{1, 0, -1\}$ is the curvature parameter. As such, the values of k determine a universe positively curved and closed, flat, or negatively curved and open. r is the radial Gaussian coordinate and t is the universal cosmic time.

Cosmological models, derived from (1.1) are called FRW models. For the case $\mathcal{R} = \text{const.}$, we face the models of the dS family. Generally, it is assumed that in this family, a crowd of points is moving apart from each point of these homogeneous spaces. The metric (1.1) should carry the information for these motions if the variables in these metrics are comoving with the points of the crowd. For $\mathcal{R} = \mathcal{R}(t)$, a variety of expanding, contracting, or pulsating universes can be derived. We use the tetrad representation, i.e., local rods and clocks for measuring physical quantities and also use the original Minkowski notation $x^4 = i(c)t$.

* e-mail: arg@aon.at; home page: http://arg.or.at/

2. ANOTHER VIEW ON THE FRW MODELS

We change the notation of (1.1) for a better understanding of the geometrical background of the problem. Instead of the Gauss coordinate $r = \sin\eta$, we use for $k = 1$ the relations $\mathcal{R} = \mathcal{K}\mathcal{R}_0$, $r = \mathcal{R}_0 \sin\eta$, $\mathcal{R}_0 = \text{const.}$ and obtain the following with $\mathcal{R}_0 r \to r$ and $\mathcal{K}$ as scale factor

$$ds^2 = \mathcal{K}^2 \left(\frac{1}{1 - \frac{r^2}{\mathcal{R}_0^2}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 \right) - dt^2. \tag{2.1}$$

Facing the metric (2.1), it is noticeable that the space-like part of the metric has the form of a metric on a hypersphere. In this case, the variables r, ϑ, φ are non-comoving coordinates. To verify this, we perform a further transformation with $dr = \mathcal{R}_0 \cos\eta d\eta$ and obtain

$$ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \sin^2\eta d\vartheta^2 + \mathcal{R}^2 \sin^2\eta \sin^2\vartheta d\varphi^2 - dt^2. \tag{2.2}$$

For $k = -1$, we substitute $r = \mathcal{R}\,\mathrm{sh}\eta$ and obtain

$$ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \mathrm{sh}^2\eta d\vartheta^2 + \mathcal{R}^2 \mathrm{sh}^2\eta \sin^2\vartheta d\varphi^2 - dt^2. \tag{2.3}$$

Finally, for $k = 0$, we obtain a flat metric. Evidently, the spatial parts for the metrics (2.2) and (2.3) are the line elements of a three-dimensional hypersphere, embedded into a four-dimensional flat space. For $k = 1$, the radius of the hypersphere is real; for $k = -1$, the radius is imaginary. Thus, $\eta, \vartheta,$ and φ are *non-comoving* coordinates on the spheres. The time t is a 'cylindrical' variable.

Looking for a possible interpretation of the metrics, we note that the lapse function of both metrics is $g_{44} = 1$. It follows that no gravitational force can be derived for both metrics. Consequently, both spaces are free of gravitation and the models cannot be used for physical interpretation, or the lapse function indicates that the observers in these spaces are in free fall and t is a *comoving coordinate*.

In this case, we realize that the FRW metrics are built of two kinds[1] of metric coefficients: one referring to comoving and the others to non-comoving coordinates. Thus, we call these FRW metrics *hybrid metrics*.

Florides [1] interpreted the variables of (1.1) as comoving coordinates and searched for metrics in non-comoving coordinates for the case of $\mathcal{R} = \text{const.}$. In our papers [2][3], we reformulated the procedures in tetrad calculus. Consequently, we were able to complement the calculations with Lorentz transformations or pseudo-rotations. We found expressions that we did not believe in. Investigations concerning this problem were made by Mitra [4][5] and Lachieze-Rey[2] [6]. Gautreau [7] discussed only flat models ($k = 0$). The question that arises is why one must deduce a metric in static coordinates from an FRW metric, although several models were presented by the original authors with embeddings, i.e., although the corresponding metrics are given in non-comoving coordinates.

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[1] We recall the theorem of Pythagoras: $a^2 + b^2 = c^2$. If the sides a and b are attributed to different triangles, we obtain some ominous c.

[2] They start with an FRW metric with $k = -1$ to obtain the dS model. After some calculations, they obtain a static metric that evidently differs from the well-known dS solution.

3. LEMAÎTRE METRICS

To study the problem in more detail, we investigate the dS-family consisting of the dS, AdS, Lanczos, and Lanczos-like models. For all these models, $\mathcal{R}_0 = \text{const.}$ and embeddings exist, as listed in the Appendix (A1–A4). The line elements of the Lanczos [8] and Lanczos-like models have the hybrid form of (2.2) or (2.3). They are less useful in a physical application.

From the embedding, we deduce the line elements of the dS model [9]-[12]

$$\text{(I B)} \qquad ds^2 = \mathcal{R}_0^2 d\eta^2 + \mathcal{R}_0^2 \sin^2\eta d\vartheta^2 + \mathcal{R}_0^2 \sin^2\eta \sin^2\vartheta d\varphi^2 + \mathcal{R}_0^2 \cos^2\eta di\psi^2 . \qquad (3.1)$$

The radius of the sphere in the three-dimensional spherical space is $r^2 = x^\alpha x^\alpha,\ \alpha = 1,2,3$, and $\mathcal{R}_0 di\psi = idt$ is the definition of the coordinate-time interval. With $r = \mathcal{R}_0 \sin\eta$, the line element can be written as

$$\text{(I B')} \qquad ds^2 = \frac{1}{\cos^2\eta} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \cos^2\eta dt^2 \qquad (3.2)$$

or as

$$\text{(I B'')} \qquad ds^2 = \frac{1}{1-\dfrac{r^2}{\mathcal{R}_0^2}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \left(1-\frac{r^2}{\mathcal{R}_0^2}\right) dt^2 . \qquad (3.3)$$

We remember that de Sitter has described a static cosmos by (3.2). The so-called expanding version was added later. We note that the de Sitter model is based on a time-independent geometric framework. The coordinate system used here is the *non-comoving* one.

For the AdS model, we get from the embedding the metric in non-comoving coordinates

$$\text{(II B)} \qquad ds^2 = \mathcal{R}_0^2 d\eta^2 + \mathcal{R}_0^2 \mathrm{sh}^2\eta d\vartheta^2 + \mathcal{R}_0^2 \mathrm{sh}^2\eta \sin^2\vartheta d\varphi^2 + \mathcal{R}_0^2 \mathrm{ch}^2\eta di\psi^2 , \qquad (3.4)$$

$$\text{(II B')} \qquad ds^2 = \frac{1}{\mathrm{ch}^2\eta} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \mathrm{ch}^2\eta dt^2 , \qquad (3.5)$$

$$\text{(II B'')} \qquad ds^2 = \frac{1}{1+\dfrac{r^2}{\mathcal{R}_0^2}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \left(1+\frac{r^2}{\mathcal{R}_0^2}\right) dt^2 \qquad (3.6)$$

having applied $r = \mathcal{R}_0 \mathrm{sh}\eta$. Here, (I B'') and (II B'') are written in the canonical form. From these, it is possible to read the type of curvature, i.e., $k = 1$ for the dS and $k = -1$ for the AdS model. Gravitational forces emerge in both the dS and AdS models. They can be explained as forces acting on a cloud of observers drifting apart from arbitrary points on the hyperspheres.

For both models, we have found Lemaître transformations [13] for comoving systems and obtained following metrics in *Lemaître* form*:*

$$\text{(I A)} \qquad ds^2 = \mathcal{K}^2 \left(dr'^2 + r'^2 d\vartheta^2 + r'^2 \sin^2\vartheta d\varphi^2\right) - dt'^2 , \qquad (3.7)$$

$$\text{(II A)} \qquad ds^2 = \mathcal{K}^2 \left(dr'^2 + r'^2 d\vartheta^2 + r'^2 \sin^2\vartheta d\varphi^2\right) - dt'^2 . \qquad (3.8)$$

Here, $\mathrm{K}=\mathrm{K}(t')$ is the time-dependent scale factor, describing the expansion of an assumed particle cloud in comoving coordinates and t' is the universal cosmological time, identical to the proper time of the drifting particles. Although the line elements are formally identical, they differ in terms of the definitions of r' and K. Evidently, the metrics (A) are of type $k=0$, which is typical for a flat space. This does not imply that curved spaces with $k=1$ and $k=-1$ can be transformed into a *globally flat* spaces, but that they can be transformed only into a *locally flat* space, according to Einstein's elevator principle [14][15][16]. The lapse functions in (I A) and also in (II A) are $g_{4'4'}=1$ and indicate that a cloud of particles moves in free fall in the universes. No gravitational forces can be deduced from the lapse functions.

Both models have been treated in [13] in more detail. There, we have extended the dS and AdS models to expanding models and have shown that the recession velocities of the models are physical velocities and not coordinate velocities and thus cannot be explained by expansion effects.

Since the AdS model is open and infinite, admits superluminal velocities, and its geometrical structure is less applicable, we concentrate ourselves only on the dS model. We work out the very structures that a physically interpretable model should have.

4. THE BASIC STRUCTURE FOR LEMAÎTRE MODELS

We call the models having metrics of type (A) and (B) Lemaître models to distinguish them from the FRW models that expose different geometrical structures. For further processing, it is necessary to exclusively use tetrads and Ricci-rotation coefficients. The coordinate way of writing and the use of Christoffel symbols do not exhibit the desired geometrical and physical structures. In the worst case, they produce mathematical artifacts. Indeed, this was seen in a paper by Melia [17]. Concerning this problem, we [18] compared the Ricci-rotation coefficients with the Christoffel symbols and showed that the Ricci-rotation coefficients describe the curvatures of slices of a sphere, which is the basis of the model, while the Christoffel symbols are a collection of angular functions and a relation to geometrical objects is missing.

To start with, we decompose the Ricci-rotation coefficients as follows:

$$A_{mn}{}^{s}=B_{mn}{}^{s}+C_{mn}{}^{s}+U_{mn}{}^{s}. \qquad (4.1)$$

We perform a further decomposition

$$B_{mn}{}^{s}=b_m B_n b^s-b_m b_n B^s, \quad C_{mn}{}^{s}=c_m C_n c^s-c_m c_n C^s, \quad U_{mn}{}^{s}=u_m U_n u^n-u_m u_n U^s, \qquad (4.2)$$

where

$$b_m=\{0,1,0,0\}, \quad c_m=\{0,0,1,0\}, \quad u_m=\{0,0,0,1\} \qquad (4.3)$$

are the unit vectors in the second, third, and fourth directions of the space. From (3.2), we read the tetrads

$$\overset{1}{e}_1=\frac{1}{\cos\eta}, \quad \overset{2}{e}_2=r, \quad \overset{3}{e}_3=r\sin\vartheta, \quad \overset{4}{e}_4=\cos\eta \qquad (4.4)$$

and we calculate the field strengths from the Ricci-rotation coefficients[3]. The lateral field quantiles are

$$B_m = \left\{\frac{1}{r}\cos\eta, 0, 0, 0\right\}, \quad C_m = \left\{\frac{1}{r}\cos\eta, \frac{1}{r}\cot\vartheta, 0, 0\right\}. \tag{4.5}$$

They describe the curvatures[4] of the greater circles and parallels of the hypersphere. From the lapse function, one obtains the gravitational force[5]

$$U_m = \left\{-\frac{1}{\mathcal{R}_0}\tan\eta, 0, 0, 0\right\}, \tag{4.6}$$

which acts in the radial direction. This force can be interpreted in such a manner that it drives points on the hypersphere from an arbitrary point in all radial directions. One reads the velocities of these points from the lapse function of (3.3) as

$$v = \frac{r}{\mathcal{R}_0} = \sin\eta. \tag{4.7}$$

Thus, they are geometrically defined. The associated Lorentz factor is

$$\alpha = \frac{1}{\cos\eta} = \frac{1}{\sqrt{1 - r^2/\mathcal{R}_0^2}}. \tag{4.8}$$

The matrix of the Lorentz transformation is noted in the Appendix (A5).

We want to know how observers experience forces if they are comoving with the drifting points. For this purpose, we have to apply the inhomogeneous transformation law of the Ricci-rotation coefficients [19]:

$$'A_{m'n'}{}^{s'} = L_{m'n's}^{m\,n\,s'} A_{mn}{}^{s} + L_{s}^{s'} L_{n'|m'}^{s}. \tag{4.9}$$

Primes signify quantities and indices in the comoving system. After applying some algebra, we obtain the following:

$$\boxed{\begin{array}{lll} B_m = \left\{\frac{1}{r}\cos\eta, 0, 0, 0\right\} & \to & B_{m'} = \left\{\frac{1}{r}, 0, 0, -\frac{i}{\mathcal{R}_0}\right\} \\ C_m = \left\{\frac{1}{r}\cos\eta, \frac{1}{r}\cot\vartheta, 0, 0\right\} & \to & C_{m'} = \left\{\frac{1}{r}, \frac{1}{r}\cot\vartheta, 0, -\frac{i}{\mathcal{R}_0}\right\} \\ U_m = \left\{-\frac{1}{\mathcal{R}_0}\tan\eta, 0, 0, 0\right\} & \to & 'U_{m'} = \left\{0, 0, 0, -\frac{i}{\mathcal{R}_0}\right\} \end{array}}. \tag{4.10}$$

Evidently, the spatial parts of the lateral field quantities appear to be flat in the new system

$$B_{\alpha'} = \left\{\frac{1}{r}, 0, 0\right\}, \quad C_{\alpha'} = \left\{\frac{1}{r}, \frac{1}{r}\cot\vartheta, 0\right\}, \quad \alpha = 1, 2, 3. \tag{4.11}$$

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[3] More details can be found in our monographs [19].

[4] In a local five-dimensional tetrad system, these quantities would have the following components:

$$B_a = \left\{\frac{1}{r}\sin\eta, \frac{1}{r}\cos\eta, 0, 0, 0\right\}, \; C_a = \left\{\frac{1}{r}\sin\eta, \frac{1}{r}\cos\eta, \frac{1}{r}\cot\vartheta, 0, 0\right\} \text{with} \sqrt{B^a B_a} = \frac{1}{r}, \; \sqrt{C^a C_a} = \frac{1}{r\sin\vartheta}, \; a = 0, 1, \ldots, 4.$$

[5] U is the geometrical force; -U the physical force.

An observer in free fall experiences space as *locally flat*, which is in agreement with Einstein's cosmic elevator [14][15]. Moreover, the quantity $'U$ does not have any radial component, indicating that no gravitational forces are acting on the freely falling observer. In addition, all three quantities contain a fourth component, the tidal forces. They describe the expansion of a volume element that encloses the diverging points in the dS cosmos. Indeed, the expansion scalar is

$$'u^{m'}{}_{||m'} = 'A_{m's'}{}^{m'}\,'u^{s'} = -3\frac{i}{\mathcal{R}_0^2}, \quad \frac{D'u^{4'}}{Dt'} = \frac{3}{\mathcal{R}_0^2}, \quad 'u_{m'} = \{0,0,0,1\}, \tag{4.12}$$

where $'u_{m'}$ is the comoving observer.

Eqs. (4.10) are the fundamental equations for a cosmological model based on a pseudo-hypersphere with a constant curvature. All static models with drifting observers have to have this structure if spherical coordinates are used. We recall that even the non-physical AdS model shows this structure. Certainly, it is a Lemaître model, too. In contrast, we show in the Appendix (A6), (A7) that other models, derived from the FRW metric with a Florides transformation, do not have this structure. Both Lanczos models contain non-flat components in the space-like part of the comoving metrics although the cosmic time indicates free fall.

Now, we will calculate the 4-beine for the comoving system. The transformation from the non-comoving to the comoving system is

$$\overset{m'}{e}{}_{i'} = L^{m'}{}_{m}\overset{m}{e}{}_{i}\Lambda^{i}{}_{i'}. \tag{4.13}$$

The Lorentz transformation L is given by (A5) and the Lemaître coordinate transformation [20][21] Λ by (A8). Finally, we obtain the comoving reference system as follows:

$$\overset{1'}{e}{}_{1'} = \mathcal{K}, \quad \overset{2'}{e}{}_{2'} = r, \quad \overset{3'}{e}{}_{3'} = r\sin\vartheta, \quad \overset{4'}{e}{}_{4'} = 1. \tag{4.14}$$

Here is $\mathcal{K} = e^{t'/\mathcal{R}_0}$, $r = \mathcal{K}r'$, r' is the comoving radial coordinate, and t' is the universal cosmological time. For the metric, one gets the well-known Lemaître form of

$$ds^2 = \mathcal{K}^2\left(dr'^2 + r'^2\,d\vartheta^2 + r'^2\sin^2\vartheta d\varphi^2\right) - dt'^2, \tag{4.15}$$

being a special case of the FRW metrics for $k = 0$ and describing a *locally flat* space. From the above metric, one can again derive the equations (4.10) for the comoving system.

5. TIME-DEPENDENT MODELS

The simplest way to get a time-dependent model expanding in free fall from a static model based on a pseudo-hypersphere is to omit the condition $\mathcal{R} = \text{const.}$. Now, $\mathcal{R}$ is the radius of the expanding hypersphere. Starting with an FRW metric, we have to restrict ourselves to the case $k = 0$, i.e., to a Lemaître metric – a metric describing observers in free fall.

The requirements for the seed metric of an expanding model are perfectly fulfilled by the dS model. We derived such an expanding model [22] and called it the Subluminal Model. This expanding model is presented as a series of self-similar dS models, and the velocities of the drifting particles of the dS model are adopted as the recession velocities of the galaxies.

Another non-hybrid model was presented by Melia[6] and called the R_h=ct model. It starts with the metric of type (4.15). But the specification $k=0$ is interpreted by Melia as a property of a *globally flat* space being infinite. But it can be shown that the R_h=ct model can be reformulated in such a manner that it is identical to our Subluminal Model. Both models, the R_h =ct model and the Subluminal Model, and also the unphysical expanding version of the static AdS model, are exact solutions of Einstein's field equations. The FRW models, particularly the standard model, are commonly not exact solutions[7] of Einstein's field equations and are hybrid models.

In the standard model, the pressure tacitly is put in by hand. This has the consequence of several parameters – the Ωs and the deceleration parameter – having to be determined by data of astrophysicists. In contrast, only one parameter has to be fixed by astronomical observation for the exact non-hybrid models. This is the time-dependent scale factor $\mathcal{K}$ or the age of the universe, respectively. Referring to our Subluminal Model, it concerns the radius of the pseudo-hypersphere $\mathcal{R}(t')=\mathcal{K}(t')\mathcal{R}_0$, $\mathcal{R}_0$ being a constant and $\dot{\mathcal{K}}=1$, $\ddot{\mathcal{K}}=0$. The expansion is linear and no inflation occurs. Melia has shown in several papers that observational data have best fit to his R_h=ct model. For the same reason, our Subluminal Model is favored.

Moreover, in the hybrid models, second derivatives of the scale factor enter into the Friedman equation, which indicate acceleration. This is in contradiction to Einstein's elevator principle, which demands that no acceleration should appear in freely falling systems.

Lastly, we face a problem that seems to be important for some authors. They raise the question of whether a non-comoving coordinate system can be determined from the FRW ansatz also for expanding models. Here, we note a paper of Melia [23] and Mitra [4]. Such desired coordinates are sometimes called Schwarzschild coordinates or curvature coordinates. We doubt that it could be possible to derive such coordinates for expanding models, as for a time-dependent radius of the pseudo-hypersphere, a new quantity enters the theory: $\mathcal{F}_{4'}=\frac{1}{\mathcal{R}}\mathcal{R}_{|4'}=\frac{1}{\mathcal{K}}\mathcal{K}_{|4'}$. This describes the expansion of a volume element. As we do with a genuine expansion, a fictive non-comoving observer should also recognize this enlargement of volume elements. With a Lorentz transformation to a non-comoving system, one obtains the quantity $\mathcal{F}_m=L_m^{4'}\mathcal{F}_{4'}$ accompanying the gravitational force emerging in the non-comoving system. This quantity cannot be absorbed by the non-linear term of the transformation law of the Ricci-rotation coefficients. The total expression is not a gradient and not integrable; the grounds are quite reasonable. The Friedman equation is a result of two systems of differential equations – Einstein's field equations and the Bianchi identities. The former describe the curvature, while the latter describe the change of the curvature of the pseudo-hypersphere. But the lapse function is a result of Einstein's field equations alone and cannot be determined by the Bianchi identities. Indeed, one would get a lapse function in switching off the expansion, i.e., to do without the Bianchi identities.

Another problem occupies the cosmologists – the Milne cosmos [24][25]. The model of Milne is based on the principles of special relativity. In a flat space, there is first concentrated matter, which spreads into all directions after an initial process. The redshift of the light emanating from the receding stars has a kinematic origin and is explained by the Doppler effect. Interestingly, the Milne cosmos is also regarded as a special case of the Friedman cosmos, with the curvature parameter being $k=-1$. With a suitable

[6] Quotation of the papers by Melia and his coworkers can be found in [22].

[7] Several authors are working with equations that have actually not been solved.

coordinate transformation, one can bring the metric into a flat form in order to obtain the original description of the Milne cosmos. We want to highlight this process critically [26].

Specializing the Friedman metric with $\mathcal{R}^{\bullet} = 1$ and taking $\mathcal{R} = t$, we obtain the following with the coordinate transformation introduced by Walker [26]:

$$r' = t\,\mathrm{sh}\eta, \quad t' = t\,\mathrm{ch}\eta\,. \tag{5.1}$$

One thus obtains from the $k = -1$ metric a flat metric as expected for the original Milne metric:

$$ds^2 = dr'^2 + r'^2 d\vartheta^2 + r'^2 \sin^2\vartheta d\varphi^2 - dt'^2\,. \tag{5.2}$$

Although the specification $\mathcal{R}^{\bullet} = 1$ is possible, one has confused the curvature radius $\mathcal{R}$ with the time t. This gives the impression that a space with a negative curvature can be transformed into a flat space by means of a coordinate transformation. This violates the principle of covariance.

If we rewrite the transformation (5.1) as

$$x^{1'} = \mathcal{R}\,\mathrm{sh}\eta, \quad x^{0'} = i\mathcal{R}\,\mathrm{ch}\eta\,, \tag{5.3}$$

we recognize that new coordinates $x^{0'}, x^{1'}$ are the Cartesian coordinates of the five-dimensional embedding space of the $k = -1$ model. The complete embedding is noted in the Appendix (A9). It describes a hypersphere complemented with an additional ‘cylindrical’ time dimension. Evidently, the model is hybrid, which also can be read from the familiar Friedman metric. In one of his papers, Melia also could not resist the temptation to use the scale factor as the time variable.

The preceding discussion shows that it is entirely justified to critically shed light on cosmological models.

6. CONCLUSIONS

We have reinvestigated the FRW models and shown that the cases for positive and negative curvatures are hybrid. This means that the metrics contain non-flat components in comoving coordinates although the lapse functions have the value 1, which is typical for freely falling systems in expanding universes and where the time is the universal cosmic time.

Only for the case $k = 0$ do field quantities emerge which can be interpreted geometrically correctly and a physical explanation can be provided for them. Since the structures and transformations have been elaborated by Lemaître for the Schwarzschild model and dS model, we call the metrics Lemaître metrics to distinguish them from the hybrid FRW metrics. The Lemaître metrics describe freely falling observers in the Schwarzschild field or in static cosmological models. The latter can be further generalized to expanding cosmological models.

7. MATHEMATICAL APPENDIX

The embedding of the dS model is

$$\begin{aligned} x^3 &= \mathcal{R}_0 \sin\eta \sin\vartheta \sin\varphi \\ x^2 &= \mathcal{R}_0 \sin\eta \sin\vartheta \cos\varphi \\ x^1 &= \mathcal{R}_0 \sin\eta \cos\vartheta \\ x^4 &= \mathcal{R}_0 \cos\eta \sin i\psi \\ x^0 &= \mathcal{R}_0 \cos\eta \cos i\psi \end{aligned} \quad . \tag{A1}$$

The embedding of the AdS model is

$$\begin{aligned} x^3 &= \mathcal{R}_0 \mathrm{sh}\eta \sin\vartheta \sin\varphi \\ x^2 &= \mathcal{R}_0 \mathrm{sh}\eta \sin\vartheta \cos\varphi \\ x^1 &= \mathcal{R}_0 \mathrm{sh}\eta \cos\vartheta \\ x^4 &= i\mathcal{R}_0 \mathrm{ch}\eta \sin\psi \\ x^0 &= i\mathcal{R}_0 \mathrm{ch}\eta \cos\psi \end{aligned} \quad . \tag{A2}$$

The embedding of the Lanczos model is

$$\begin{aligned} x^3 &= \mathcal{R}_0 \cos i\psi \sin\eta \sin\vartheta \sin\varphi \\ x^2 &= \mathcal{R}_0 \cos i\psi \sin\eta \sin\vartheta \cos\varphi \\ x^1 &= \mathcal{R}_0 \cos i\psi \sin\eta \cos\vartheta \\ x^0 &= \mathcal{R}_0 \cos i\psi \cos\eta \\ x^4 &= \mathcal{R}_0 \sin i\psi \end{aligned} \quad . \tag{A3}$$

The embedding of the Lanczos-like model is

$$\begin{aligned} x^3 &= \mathcal{R}_0 \mathrm{sh}\psi\, \mathrm{sh}\eta \sin\vartheta \sin\varphi \\ x^2 &= \mathcal{R}_0 \mathrm{sh}\psi\, \mathrm{sh}\eta \sin\vartheta \cos\varphi \\ x^1 &= \mathcal{R}_0 \mathrm{sh}\psi\, \mathrm{sh}\eta \cos\vartheta \\ x^4 &= i\mathcal{R}_0 \mathrm{sh}\psi\, \mathrm{ch}\eta \\ x^0 &= \mathcal{R}_0 \mathrm{ch}\psi \end{aligned} \quad . \tag{A4}$$

The Lorentz transformation to a non-comoving system is

$$L^{m}_{m'} = \begin{pmatrix} \frac{1}{\cos\eta} & & & -i\tan\eta \\ & 1 & & \\ & & 1 & \\ i\tan\eta & & & \frac{1}{\cos\eta} \end{pmatrix} = \begin{pmatrix} \alpha & & & -i\alpha v \\ & 1 & & \\ & & 1 & \\ i\alpha v & & & \alpha \end{pmatrix} \tag{A5}$$

Typical field quantities for hybrid models are

$$B_m=\left\{\frac{1}{r}\cos\eta,0,0,0\right\},\quad C_m=\left\{\frac{1}{r}\cos\eta,\frac{1}{r}\cot\vartheta,0,0\right\},\quad U_m=\left\{-\frac{1}{\mathcal{R}_0}\tan\eta,0,0,0\right\},$$

$$B_{m'}=\left\{\frac{1}{r}\cos\eta',0,0,-\frac{i}{\mathcal{R}_0}\mathrm{th}\psi'\right\},\quad C_{m'}=\left\{\frac{1}{r}\cos\eta',\frac{1}{r}\cot\vartheta,0,-\frac{i}{\mathcal{R}_0}\mathrm{th}\psi'\right\},\tag{A6}$$

$$'U_{m'}=\left\{0,0,0,-\frac{i}{\mathcal{R}_0}\mathrm{th}\psi'\right\},$$

and

$$B_m=\left\{\frac{1}{r}\sqrt{1-\mathrm{sh}^2\eta},0,0,0\right\},\quad C_m=\left\{\frac{1}{r}\sqrt{1-\mathrm{sh}^2\eta},\frac{1}{r}\cot\vartheta,0,0\right\},$$

$$U_m=\left\{-\frac{1}{\mathcal{R}_0}\frac{\mathrm{sh}\eta}{\sqrt{1-\mathrm{sh}^2\eta}},0,0,0\right\},\tag{A7}$$

$$B_{m'}=\left\{\frac{1}{r}\cos\eta',0,0,-\frac{i}{\mathcal{R}_0}\mathrm{cth}\psi'\right\},\quad C_{m'}=\left\{\frac{1}{r}\cos\eta',\frac{1}{r}\cot\vartheta,0,-\frac{i}{\mathcal{R}_0}\mathrm{cth}\psi'\right\},$$

$$'U_{m'}=\left\{0,0,0,-\frac{i}{\mathcal{R}_0}\mathrm{cth}\psi'\right\}.$$

The Lemaître coordinate transformation for the dS model is

$$\Lambda^{i}_{i'}=\begin{pmatrix}\mathcal{K} & & & -i\sin\eta\\ & 1 & & \\ & & 1 & \\ i\mathcal{K}\dfrac{\sin\eta}{\cos^2\eta} & & & \dfrac{1}{\cos^2\eta}\end{pmatrix},\qquad \Lambda^{i'}_{i}=\begin{pmatrix}\dfrac{1}{\mathcal{K}\cos^2\eta} & & & \dfrac{i\sin\eta}{\mathcal{K}}\\ & 1 & & \\ & & 1 & \\ -i\dfrac{\sin\eta}{\cos^2\eta} & & & 1\end{pmatrix}.\tag{A8}$$

The embedding of the $k=-1$ ‘Milne’ model is

$$\begin{aligned}x^{3'}&=i\mathcal{R}\sin i\eta\sin\vartheta\sin\varphi=\mathcal{R}\,\mathrm{sh}\eta\sin\vartheta\sin\varphi\\ x^{2'}&=i\mathcal{R}\sin i\eta\sin\vartheta\cos\varphi=\mathcal{R}\,\mathrm{sh}\eta\sin\vartheta\cos\varphi\\ x^{1'}&=i\mathcal{R}\sin i\eta\cos\vartheta=\mathcal{R}\,\mathrm{sh}\eta\cos\vartheta\\ x^{0'}&=i\mathcal{R}\cos i\eta=i\mathcal{R}\,\mathrm{ch}\eta\\ x^{4'}&=it\end{aligned}\quad .\tag{A9}$$

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