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REMARKS ON THE COMPLETE SCHWARZSCHILD SOLUTION

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Contents

We will show that the Schwarzschild interior and exterior solution can be represented by a common formal system if one uses the methodology of 5-dimensional embedding. Black holes are excluded from the outset. Moreover, it is not possible to approach the event horizon. The interior part of the solution covers that critical region.

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1. INTRODUCTION

To obtain a common representation of both solutions of the Schwarzschild theory, the interior and exterior, we have to use a common co-ordinate system for both solutions. The appropriate co-ordinate system for this aim is the co-ordinate system of the interior solution, which will be extended to the exterior solution. While for the interior part the origin of the co-ordinate system is fixed, the origin of the exterior part is moving on the co-ordinate axis of the extra dimension in the flat embedding space. In former papers [1, 2] we have shown that five dimensions for embedding the Schwarzschild geometry are sufficient. However, one has to use six variables. Thus, the theorems of Kasner and Eisenhart are not violated. For more details, we refer to our papers.

2. THE CO-ORDINATE SYSTEM

We use a Cartesian co-ordinate system in the 5-dimensional flat embedding space as shown in Fig. 1. For the sake of simplicity we suppress all co-ordinates except the extra co-ordinate R, also labeled by $x^{0'}$ and the standard Schwarzschild co-ordinate r, also labeled $x^{1'}$. The orientation of R is opposite to the orientation of the extra co-ordinate R of the exterior solution in our former paper [1]. The orientation of the polar angle η is ccw with respect to the local extra co- ordinate x^{0}.

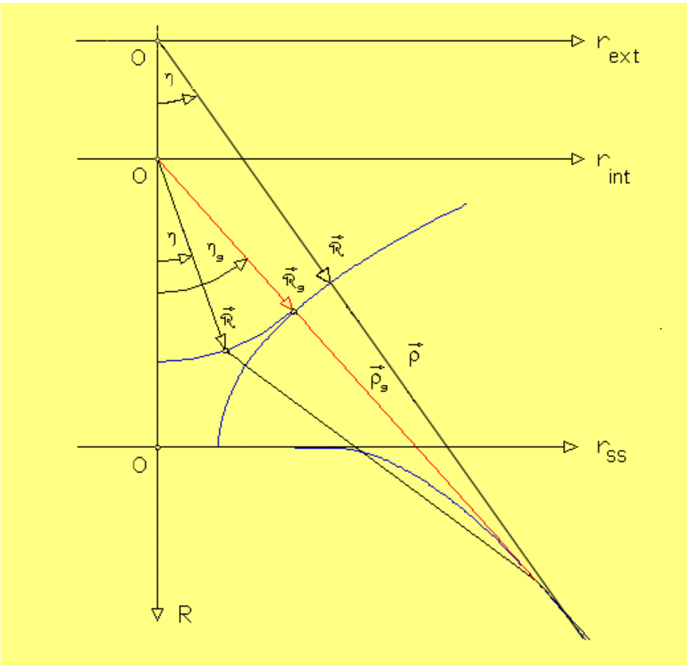

Fig. 1

For the interior part of the solution, the origin of the co-ordinate system is fixed on the symmetry axis of the system. The radius vector $\vec{\mathcal{R}}$ points to the cap of the sphere which represents the space-like part of the interior solution. The aperture angle of the cap is called η_g. $\vec{\mathcal{R}}_g$ is pointing to the boundary surface of the two solutions. In contrast, the origin of the exterior part is moving on the symmetry axis. The polar angle η is a function of the co-ordinate r. The prolongation of $\vec{\mathcal{R}}$ is the curvature vector $\vec{\rho}$ of the Schwarzschild parabola. The tip of $\vec{\rho}$ is located on the Neil parabola. If one follows the motion of a point to the center of gravitation the tip of the vector $\vec{\mathcal{R}}$ moves on the Schwarzschild parabola to the boundary surface, and then on a circle of the cap of the sphere to the symmetry axis. The tail of $\vec{\rho}$ moves to the boundary surface, as well. Then the curvature vector is called $\vec{\rho}_g$. Hereupon an auxiliary vector moves on the cap synced with $\vec{\mathcal{R}}$. This auxiliary vector has already been treated in [2] and it will not be discussed in this paper. On the boundary surface the tip of $\vec{\rho}$ is fixed on Neil's parabola and coincides with the fixed tip of the auxiliary vector.

In accordance with the new co-ordinate system, one has to take the negative root of the Schwarzschild parabola and the positive root of the Neil parabola

$$R = -\sqrt{8M(r-2M)}, \quad \bar{R} = +\sqrt{\frac{2}{M}\left(\frac{\bar{r}}{3} - 2M\right)^3} . \tag{2.1}$$

Since the polar angle η is the angle of ascent of the Schwarzschild parabola as well, one obtains from (2.1)

$$dR = -\tan\eta dr, \quad d\bar{R} = \cot\eta d\bar{r} , \tag{2.2}$$

where $\{x^{0'}, x^{1'}\} = \{R, r\}$ and $\{\bar{x}^{0'}, \bar{x}^{1'}\} = \{\bar{R}, \bar{r}\}$ are the Cartesian co-ordinates of the points of the Schwarzschild parabola and the Neil parabola. The velocity of a freely falling object is

$$v = -\sin\eta = -\sqrt{2M/r} . \tag{2.3}$$

The redshift factor is

$$a = \cos\eta = \sqrt{1-2M/r} . \tag{2.4}$$

The motion of the curvature vectors of the two parts of the solution is substantial and explains most of the geometric properties of the model. In Fig. 2 and in Fig. 3 we illustrate the curvature vectors and their differentials evoked by this motion. If

$$\{dx^{0'}, dx^{1'}\} = \{dR, dr\}, \quad \{d\bar{x}^{0'}, d\bar{x}^{1'}\} = \{d\bar{R}, d\bar{r}\} \tag{2.5}$$

are the changes of the curvature vectors of the Schwarzschild parabola and of the Neil parabola, expressed in Cartesian co-ordinates of the embedding space, then one obtains with the rotation matrix

$$D^a_{a'} = \begin{pmatrix} \cos\eta & \sin\eta \\ -\sin\eta & \cos\eta \end{pmatrix}, \quad a = 0,1 \tag{2.6}$$

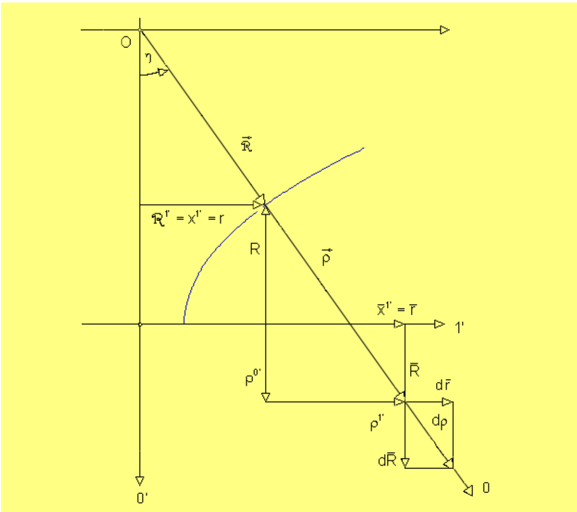

Fig. 2

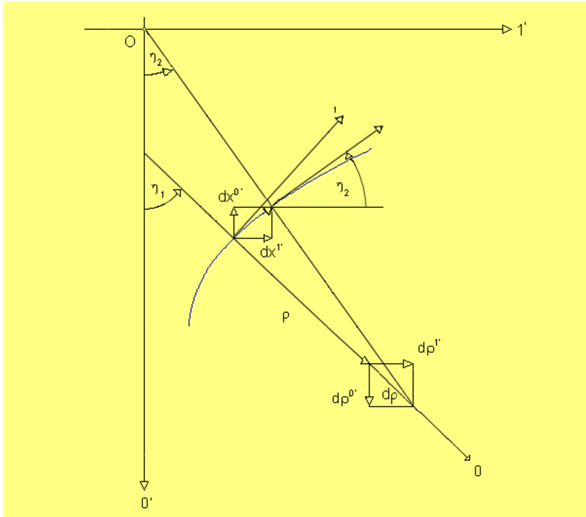

Fig. 3

and with the help of (2.2) the components of the differentials of both curves with respect to the local system

$$dx^a = \left\{0, \frac{1}{\cos\eta} dr\right\}, \quad d\bar{x}^a = \left\{\frac{1}{\sin\eta} d\bar{r}, 0\right\} . \tag{2.7}$$

The change of the curvature vector during its outward motion consists of two parts. The one contribution stems from the motion of the tail of the curvature vector along the Schwarzschild parabola, the other from the motion of the tip along the Neil parabola. Thus, we have

$$d\rho^a = \left\{d\bar{x}^a - dx^a\right\} = \left\{\frac{1}{\sin\eta} d\bar{r}, -\frac{1}{\cos\eta} dr\right\} . \tag{2.8}$$

Since the curvature vector of the Schwarzschild parabola has the value $\rho = \sqrt{2r^3/M}$ one finds once more by differentiation of this expression and with the help of (2.3) $d\rho = d\rho^0 = d\bar{r}/\sin\eta$, where the well-known relation $\bar{r} = 3r$ has been applied. Moreover, one likewise obtains with (2.3) $d\sin\eta = -dr/\rho = \cos\eta d\eta$. And finally,

$$dx^1 = \frac{1}{\cos\eta} dr = -\rho d\eta \ . \tag{2.9}$$

dx^1 is the positive tangent vector of the Schwarzschild parabola. The right side of (2.9) is positive as well because $d\eta = \eta_2 - \eta_1$ is negative. The reason is that the angle η decreases throughout the outward motion. Utilizing the two just-derived relations one has

$$d\rho^a = \left\{d\rho, \rho d\eta\right\} . \tag{2.10}$$

The first component thereof describes the change of ρ on the Neil parabola, the second on the Schwarzschild parabola. As a byproduct we note

$$\eta_{|1} = -\frac{1}{\rho}. \tag{2.11}$$

We frequently make use of this relation. Transforming (2.8) with the inverse matrix of (2.6) to the Cartesian co-ordinates of the embedding space, one gets by using (2.2)

$$d\rho^{a'} = \left\{ d\bar{R} - dR,\, d\bar{r} - dr \right\}, \tag{2.12}$$

a relation that one can see in ones mind's eye with Fig. 2. Bearing in mind that R is negative in the new co-ordinate system one can write the curvature vector with the use of the Cartesian co-ordinate system of the embedding space as

$$\rho^2 = \left(\bar{R} - R\right)^2 + \left(\bar{r} - r\right)^2. \tag{2.13}$$

Thereof one directly obtains (2.12). By differentiation of this relation one gets

$$d\rho^0 = d\rho = \frac{\bar{R} - R}{\rho} d\rho^{0'} + \frac{\bar{r} - r}{\rho} d\rho^{1'} \tag{2.14}$$

and by comparison with the rotation matrix (2.6)

$$\cos\eta = \frac{\bar{R} - R}{\rho}, \quad \sin\eta = \frac{\bar{r} - r}{\rho}. \tag{2.15}$$

Operating on the Schwarzschild parabola with the rotation matrix one has for the local partial derivatives

$$\partial_0 = \cos\eta \frac{\partial}{dR} + \sin\eta \frac{\partial}{dr}, \quad \partial_1 = -\sin\eta \frac{\partial}{dR} + \cos\eta \frac{\partial}{dr}. \tag{2.16}$$

Horizontal quantities are quantities situated in the horizontals of the surfaces. They are independent of R. For such quantities remains in (2.16) only

$$\partial_0 = \sin\eta \frac{\partial}{dr}, \quad \partial_1 = \cos\eta \frac{\partial}{dr}. \tag{2.17}$$

In particular, one has

$$r_{|a} = \left\{ \sin\eta, \cos\eta \right\}. \tag{2.18}$$

Heretofore only changes of ρ have been considered where the curvature vector has been constrained to the Schwarzschild parabola. To apply embeddings one has to face the environment of the embedded space and has to examine the exterior geometry. Therefore, the *prolongations* of the quantities into the embedding space are required. Although an infinitesimal prolongation would be sufficient, the model provides a co-ordinate system (Fig. 4) that enables a global prolongation. Going on upwards on the Figure to a curve parallel to the Schwarzschild parabola one lengthens the curvature vector by $d\rho$. Bearing in mind the orientation of the co-ordinate system one has, however,

$$dx^0 = -d\rho, \quad \partial_0 = -\frac{\partial}{\partial \rho}. \tag{2.19}$$

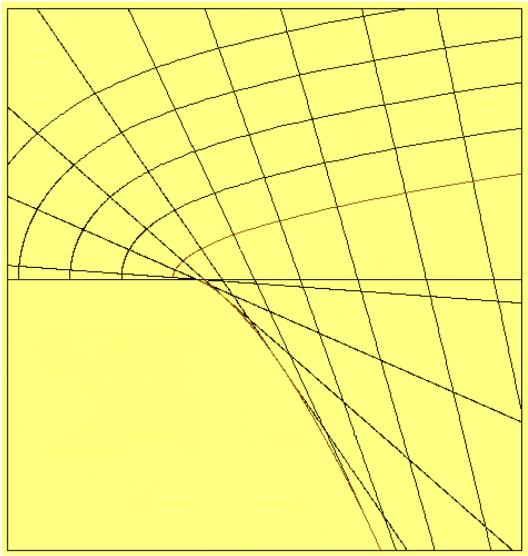

Fig. 4

In the last paragraphs the techniques necessary to set up the embedding of the Schwarzschild model have been treated in relation to the new co-ordinate system and in consideration of the unusual signs. Neither the moving origin[1] have been discussed in detail nor the common version of the two Schwarzschild solutions. We make up leeway in the next Section.

3. THE EMBEDDING

Let us envisage once more Fig. 1. The vector $\vec{\mathcal{R}}$ points from the 0'-axis of the embedding space to the Schwarzschild parabola. $\vec{\mathcal{R}}$ is normal to the parabola and has as prolongation the curvature vector. The values of these vectors have the ratio 1:2. The vector $\vec{\mathcal{R}}$ settles the local 0-axis. Its components are

$$\mathcal{R}^{a'} = \{\mathcal{R}\cos\eta, \mathcal{R}\sin\eta\}, \quad \mathcal{R}^a = \{\mathcal{R}, 0\}, \quad d\mathcal{R}^a = \{d\mathcal{R}, \mathcal{R}d\eta\}. \tag{3.1}$$

A further vector $\bar{\mathcal{R}}^a$ pointing from the origin of the local co-ordinate system to the Neil parabola is defined by

$$\bar{\mathcal{R}}^a = \mathcal{R}^a + \rho^a, \tag{3.2}$$

where the values $\mathcal{R}$ and $\bar{\mathcal{R}}$ have the ratio 1:3. We find the relation

[1]The distance from the moving origin of the standard Schwarzschild origin on the 0'-axis is $d(r) = (r + 4M)\sqrt{\frac{r}{2M} - 1}$.

$$d\rho^a = \left\{ d\bar{\mathcal{R}} - d\mathcal{R},\, \left(\bar{\mathcal{R}} - \mathcal{R}\right) d\eta \right\}. \tag{3.3}$$

The ansatz (3.1) can be used for the interior solution as well. Having left the Schwarzschild parabola, $\vec{\mathcal{R}}$ moves along the arc of a circle. The curvature vector of the Schwarzschild parabola stops at the boundary surface and is responsible for the factor 3 in the time-like arc element of the interior solution. We start with a pseudo-hyper sphere

$$\begin{aligned}
\mathcal{R}^{3'} &= \mathcal{R}\sin\eta\sin\vartheta\,\sin\varphi \\
\mathcal{R}^{2'} &= \mathcal{R}\sin\eta\sin\vartheta\cos\varphi \\
\mathcal{R}^{1'} &= \mathcal{R}\sin\eta\cos\vartheta \\
\mathcal{R}^{0'} &= \mathcal{R}\cos\eta\cos i\psi \\
\mathcal{R}^{4'} &= \mathcal{R}\cos\eta\,\sin i\psi
\end{aligned} \tag{3.4}$$

for both parts of the solution. This pseudo-hyper sphere is a component of a double surface. The field strengths and the field equations of this simple model have been discussed in [2]. Suitable projectors referring to the theory of double surfaces settle the field equations for both parts of the Schwarzschild model. The derivation of the field equations is worked out in one calculation. Firstly, we note the projectors

$$\begin{aligned}
E:&\quad \mathcal{p}_0^0 = \mathcal{p}_1^1 = \mathcal{p}_4^4 = -\frac{\mathcal{R}}{\rho}, \quad \mathcal{p}_2^2 = \mathcal{p}_3^3 = \frac{\mathcal{R}\sin\eta}{r} \\
I:&\quad \mathcal{p}_0^0 = \mathcal{p}_1^1 = \mathcal{p}_2^2 = \mathcal{p}_3^3 = 1, \quad \mathcal{p}_4^4 = -\frac{\mathcal{R}\cos\eta}{3\mathcal{R}_g\cos\eta_g - \mathcal{R}\cos\eta}
\end{aligned}\,. \tag{3.5}$$

One obtains the gravitational field strengths E_a of both parts of the solution from the Ricci-rotation coefficients

$$A_{4a}{}^4 = \mathcal{p}_4^4\,\mathcal{R}_{4a}{}^4 = -E_a. \tag{3.6}$$

If ρ_g is the curvature vector at the boundary surface of the two parts of the solution one has

$$E:\quad E_a = \left\{ \frac{1}{\rho}\frac{\cos\eta}{a_T}, -\frac{1}{\rho}\frac{\sin\eta}{a_T} \right\}, \quad I:\quad E_a = \left\{ \frac{1}{\rho_g}\frac{\cos\eta}{a_T}, -\frac{1}{\rho_g}\frac{\sin\eta}{a_T} \right\}. \tag{3.7}$$

Therein is

$$E:\quad a_T\left(x^1\right) = \cos\eta, \quad I:\quad a_T\left(x^0, x^1\right) = \left[3\mathcal{R}_g\cos\eta_g - \mathcal{R}\cos\eta\right]\frac{1}{\rho_g}. \tag{3.8}$$

Constraining the function a_T for the interior solution onto the cap of the sphere remains

$$a_T\left(x^1\right) = \frac{1}{2}\left[3\cos\eta_g - \cos\eta\right], \tag{3.9}$$

if the embedding conditions $\mathcal{R} = \mathcal{R}_g = \text{const.}$ and $\rho_g = 2\mathcal{R}_g$ are used. This is the well-known factor of the time-like arc element of the interior solution. (3.7) shows a strong similarity of the structures of both parts of the solution. One can set up a closer relation to these equations if one writes for the time-like arc element

$$dx^4 = a_\psi di\psi, \quad a_\psi = \bar{\mathcal{R}}\cos\gamma - \mathcal{R}\cos\eta. \tag{3.10}$$

If one specifies

$$\begin{aligned} E: \quad & \bar{\mathcal{R}} = 3\mathcal{R}, \quad \gamma = \eta, \quad \Rightarrow a_\psi = 2\mathcal{R}\cos\eta \\ & dx^4 = a_T dit, \quad dit = \rho di\psi, \quad a_\psi = \rho a_T \\ & \qquad , \\ I: \quad & \bar{\mathcal{R}} = 3\mathcal{R}_g, \quad \gamma = \eta_g, \Rightarrow a_\psi = 3\mathcal{R}_g\cos\eta_g - \mathcal{R}\cos\eta \\ & dx^4 = a_T dit, \quad dit = \rho_g di\psi, \quad a_\psi = \rho_g a_T \end{aligned} \tag{3.11}$$

one has demonstrated the special features of both parts of the solution. Now we consider the change of the quantity a_ψ on the Schwarzschild parabola and normal to the Schwarzschild parabola. If one writes the force of gravity as

$$E_a = -\mathcal{P}_a^b \frac{1}{a_\psi} a_{\psi,b} = -\mathcal{P}_a^b \frac{\bar{\mathcal{R}}_{,b}\cos\gamma - \mathcal{R}_{,b}\cos\eta - \bar{\mathcal{R}}\sin\gamma\,\gamma_{,b} + \mathcal{R}\sin\eta\,\eta_{,b}}{\bar{\mathcal{R}}\cos\gamma - \mathcal{R}\cos\eta} \tag{3.12}$$

one obtains a rather complicated expression

$$\begin{aligned} E_0 &= -\mathcal{P}_0^0 \frac{\bar{\mathcal{R}}_{,0}\cos\gamma - \mathcal{R}_{,0}\cos\eta}{\bar{\mathcal{R}}\cos\gamma - \mathcal{R}\cos\eta} \\ E_1 &= -\mathcal{P}_1^1 \frac{-\bar{\mathcal{R}}\sin\gamma\,\gamma_{,1} + \mathcal{R}\sin\eta\,\eta_{,1}}{\bar{\mathcal{R}}\cos\gamma - \mathcal{R}\cos\eta} \end{aligned}. \tag{3.13}$$

With the special values (3.5) and (3.11) we recover the quantities (3.7). For the development of the model, one has to evaluate the derivatives of the projectors. Above all, $\mathcal{P}_4^4$ is deduced from the ratio of the time-like metric factors a_H of the spherical geometry and a_ψ of the Schwarzschild geometry

$$\mathcal{P}_4^4 = \frac{a_H}{a_\psi}, \quad a_H = \mathcal{R}\cos\eta. \tag{3.14}$$

We note that the two variables in a_H are independent. In contrast, the quantities $\bar{\mathcal{R}}$ and $\mathcal{R}$ depend on r for the exterior part of the solution because the tip and tail of the curvature vector $\vec{\rho} = \vec{\bar{\mathcal{R}}} - \vec{\mathcal{R}}$ move on the Schwarzschild parabola and on the Neil parabola. Firstly, one has

$$\mathcal{P}_{4|a}^4 = -\frac{1}{a_H} a_{H|a} - \mathcal{P}_4^4 \frac{1}{a_\psi} a_{\psi|a}.$$

With

$$a_{H,a} = \left\{\cos\eta, -\sin\eta\right\}$$

one obtains

$$\frac{1}{a_H} a_{H|a} = \mathcal{P}^b_a E_b \ .$$

Furthermore,

$$\frac{1}{a_\psi} a_{\psi|a} = \frac{\bar{\mathcal{R}}_{|a}\cos\gamma - \mathcal{R}_{|a}\cos\eta - \bar{\mathcal{R}}\sin\gamma\gamma_{|a} + \mathcal{R}\sin\eta\eta_{|a}}{\bar{\mathcal{R}}\cos\gamma - \mathcal{R}\cos\eta} \ .$$

Apart from the term

$$\Lambda_a = \delta^1_a \frac{\bar{\mathcal{R}}_{|1}\cos\gamma - \mathcal{R}_{|1}\cos\eta}{\bar{\mathcal{R}}\cos\gamma - \mathcal{R}\cos\eta}$$

this has been calculated above. The Λ-term

$$E: \quad \Lambda_1 = \frac{1}{\rho}\rho_{|1}, \quad I: \quad \Lambda_1 = 0$$

provides the important contribution Λ_1 for the exterior solution. In contrast, it vanishes for the interior solution. In total one has

$$\mathcal{P}^4_{4|a} = -\mathcal{P}^b_a E_b + \mathcal{P}^4_4 E_a + \Lambda_a \ . \tag{3.15}$$

Finally, that results in

$$\begin{array}{ll} E: & \mathcal{P}^0_0 = \mathcal{P}^1_1 = \mathcal{P}^4_4, \quad \mathcal{P}^4_{4|0} = 0, \quad \mathcal{P}^4_{4|1} = \frac{1}{\rho}\rho_{|1} = \frac{3}{\rho}\cot\eta \\ I: & \mathcal{P}^0_0 = \mathcal{P}^1_1 = 1, \qquad \mathcal{P}^4_{4|a} = -\left(1 - \mathcal{P}^4_4\right)E_a \end{array} \ . \tag{3.16}$$

Further projectors mentioned in (3.5) and their derivatives can be calculated from the metric factors of both parts of the geometry, but in an essentially simpler way. They satisfy the relation

$$\mathcal{P}^a_b = \mathcal{R}^a{}_{|||b} \ . \tag{3.17}$$

For evaluating the field equations one needs the auxiliary quantity

$$\mathcal{P}_{[ba]}{}^g = \left(\mathcal{P}^{-1}\right)^g_f \mathcal{P}^f_{[a|||b]} \ , \tag{3.18}$$

which has only a few components for the exterior solution and vanishes for the interior solution

$$E: \quad 2\mathcal{P}_{[01]}{}^{0} = \mathcal{P}_{[41]}{}^{4} = \frac{1}{\rho}\rho_{|1} = \frac{3}{\rho}\cot\eta, \quad I: \quad \mathcal{P}_{[ba]}{}^{g} = 0 . \qquad (3.19)$$

Projecting the identically vanishing curvature tensor of the embedding space

$$\mathcal{P}^{gh}_{ab}\mathcal{R}_{ghc}{}^{d} = {}^{5}R_{abc}{}^{d} + 2\mathcal{P}_{[ba]}{}^{g}A_{gc}{}^{d} \equiv 0 \qquad (3.20)$$

onto the Schwarzschild geometry one finds the 5-dimensional Ricci

$${}^{5}R_{ab} + 2\mathcal{P}_{[ac]}{}^{d}A_{db}{}^{c} \equiv 0 \; . \qquad (3.21)$$

If one performs the dimensional reduction, this relation simplifies to

$${}^{5}R_{mn} + 2\mathcal{P}_{[mr]}{}^{s}A_{sn}{}^{r} = 0 \; . \qquad (3.22)$$

To get the 4-dimensional Ricci one has to extract the 0-components from the 5-dimensional Ricci. These 0-components we condense to

$$Z_{mn} = 2A_{[m}{}^{s}A_{s]n} . \qquad (3.23)$$

The A_{mn} are the generalized second fundamental forms of the theory of surfaces. One obtains

$${}^{4}R_{mn} + Z_{mn} + 2\mathcal{P}_{[mr]}{}^{s}A_{sn}{}^{r} = 0 . \qquad (3.24)$$

Contracting this expression and inserting it into the Einstein field equations one obtains on the right side

$$\kappa T_{mn} = \left[Z_{mn} - \frac{1}{2} Z g_{mn} \right] + 2\left[\mathcal{P}_{[mr]}{}^{s}A_{sn}{}^{r} - \frac{1}{2} g_{mn}\mathcal{P}_{[tr]}{}^{s}A_{s}{}^{tr} \right], \qquad (3.25)$$

the common stress-energy tensor of the model. It vanishes for the exterior part of the solution and provides the well-known expressions for the hydrostatic pressure and energy density for the interior part. Thus, we only need to analyze the conservation law for the interior part. With respect to the second equation of (3.19) the divergence of the stress-energy tensor simplifies to

$$\left[Z_{m}{}^{n} - \frac{1}{2} Z \delta^{n}_{m} \right]_{||n} = 2A_{<m}{}^{s}A_{[s}{}^{n}{}_{||n]>} \; . \qquad (3.26)$$

The right side of this equation corresponds to the contracted Codazzi equation and vanishes. The conservation law is satisfied in an almost general form. The pressure function contained in the stress-energy tensor has a pole. At a distinct aperture angle of the cap of the sphere which represents the space-like part of the interior solution the hydrostatic pressure becomes infinitely high. That is the reason, why the boundary surface of the two parts of the solution cannot be positioned arbitrarily. Thus, the Schwarzschild radius, occurring in the exterior solution, is outside of the physically possible range of the model. An observer is never exposed to infinitely high forces and no infalling observer can reach the velocity of light.

4. SUMMARY

We have worked out a common representation of the exterior and interior Schwarzschild solution with the help of embedding the geometry into a 5-dimensional flat space. Since the interior part of the solution must necessarily cover the critical region of the exterior solution all physical quantities are well-behaved. Furthermore, there is no room for black hole physics.

5. REFERENCES

1. Burghardt R., *New embedding of Schwarzschild geometry. I. Exterior solution* http://arg.or.at/Wpdf/W5d.pdf and Sitzungsberichte der Leibniz-Sozietät **61**, 105, 2003

2. Burghardt R., *New embedding of Schwarzschild geometry. II. Interior solution.* http://arg.or.at/Wpdf/W5I.pdf and Science Edition, Potsdam 2006, p 29.