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GÖDEL UNIVERSE AND SHEARS

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Contents

It is shown that the Gödel universe is not rotating rigidly but has a differential rotation law. Locally non-rotating observers are subject to shears. The field strengths for the shears satisfy the field equations.

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1. INTRODUCTION

In a previous paper [1] we have shown that the Gödel metric can be re-interpreted by rescaling the variables. It has turned out that the rotation of the Gödel universe is not rigid, but has a differential rotation law. The two rotational distributions sum to a constant rotational field strength, usually thought to be proportional to a constant angular velocity. As a consequence of the assumption that a differential rotation law is valid it should be possible to choose a locally non-rotating reference system in which the observers are subject to shears. In the following we will show that the Einstein field equations provide these shears.

In Sec. 2 we introduce a locally non-rotating reference system for the Gödel metric, and in Sec. 3 we evaluate the field equations for the shears.

2. THE LOCALLY NON-ROTATING SYSTEM

Gödel [2] has given the metric of a rotating universe in the form

$$ds^2 = 4a^2\left[dr^2 + dy^2 + \left(\mathrm{sh}^2 r - \mathrm{sh}^4 r\right)d\varphi^2 - 2\sqrt{2}\,\mathrm{sh}^2 r\, d\varphi\, dt - dt^2\right] , \tag{2.1}$$

satisfying the Einstein field equations for a rotating dust-filled universe. Rescaling and renaming the variables

$$a \to \mathcal{R}, \quad 2ay \to z, \quad 2at \to t, \quad 2r \to \eta, \quad r \to \chi$$

one can write this metric as

$$ds^2 = \mathcal{R}^2 d\eta^2 + dz^2 + \mathcal{R}^2 \mathrm{sh}^2\eta\, d\varphi^2 - \left[2\sqrt{2}\,\mathcal{R}\,\mathrm{sh}^2\chi\, d\varphi + dt\right]^2 \tag{2.2}$$

and with an additional substitution[1]

$$\mathrm{th}\theta = \sqrt{2}\,\mathrm{th}\chi \tag{2.3}$$

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[1] $\mathrm{sh}^2\theta = \dfrac{2\,\mathrm{sh}^2\chi}{1-\mathrm{sh}^2\chi}, \quad \mathrm{ch}^2\theta = \dfrac{\mathrm{ch}^2\chi}{1-\mathrm{sh}^2\chi}, \quad \mathrm{th}^2\theta = 2\,\mathrm{th}^2\chi, \quad \mathrm{sh}\eta = 2\,\mathrm{sh}\chi\mathrm{ch}\chi$

$$ds^2 = \mathcal{R}^2 d\eta^2 + dz^2 + \mathcal{R}^2 sh^2\eta\, d\varphi^2 - \left[\mathcal{R}\, sh\eta\, th\theta\, d\varphi + dt\right]^2 \qquad (A)\ . \tag{2.4}$$

We rearrange the metric as

$$\begin{gathered} ds^2 = \mathcal{R}^2 d\eta^2 + dz^2 + \left[2\mathcal{R}\, sh\chi\sqrt{1-sh^2\chi}\, d\varphi - \frac{\sqrt{2}sh\chi}{\sqrt{1-sh^2\chi}} dt\right]^2 - \frac{ch^2\chi}{1-sh^2\chi} dt^2 \\ ds^2 = \mathcal{R}^2 d\eta^2 + dz^2 + \left[\mathcal{R}\, sh\eta \frac{1}{ch\theta} d\varphi - sh\theta\, dt\right]^2 - ch^2\theta\, dt^2 \end{gathered} \qquad (B)\ . \tag{2.5}$$

The metric in the form (B) can be derived from (A) by a Lorentz transformation

$$L_2^{2'} = \cos i\theta, \quad L_4^{2'} = \sin i\theta, \quad L_2^{4'} = -\sin i\theta, \quad L_4^{4'} = \cos i\theta\ , \tag{2.6}$$

wherein θ is the rapidity of the motion. The matter of the universe is in relative motion with respect to new observers. The velocity of the matter relative to these observers is

$${}'u_{m'} = \{0, \sin i\theta, 0, \cos i\theta\} = \{0, i\alpha\omega\sigma, 0, \alpha\}\ . \tag{2.7}$$

We have identified the transformation parameters with the physical quantities

$$\cos i\theta = ch\theta = \alpha, \quad \sin i\theta = i\, sh\theta = i\alpha\omega\sigma\ . \tag{2.8}$$

The circular velocity of the matter and the distance from the rotation axis are

$$\omega\sigma = th\theta, \quad \sigma = \mathcal{R} sh\eta\ . \tag{2.9}$$

Evidently, the matter is rotating with the velocity of light ($th\theta = 1$) at the *cut-off radius*

$$\sigma_c = \frac{1}{\omega_c} \tag{2.10}$$

where the rapidity is infinitely large. Beyond σ_c the motion is tachyonic and gives rise to acausalities and CTCs. Cooperstock [3] has mentioned that the light cones touch the world lines in that case. Pfarr [4] discussed closed null lines and paths with velocities larger than light.

The angular velocity

$$\omega(\chi) = \frac{\sqrt{2}th\chi}{\mathcal{R}sh\eta} = \frac{1}{\sqrt{2}\mathcal{R}ch^2\chi} \tag{2.11}$$

depends on the distance to the rotation axis. Equ. (2.11) is the ansatz for the *differential rotation law*. In our previous paper [1] we have shown that the total rotational force has two contributions: a Coriolis-like one and a second, invoked by dω. Both contributions depend on the radial coordinate, but they sum to the total rotational force. This force is constant over all the space. This might pretend a constant rotational velocity.

Our aim is to show that in the local non-rotating system (system B) no Coriolis field strengths appear and that the observer experiences shears. We also show that the field strengths responsible for the shears, satisfy the field equations. Thus, we present a good argument for the assertion that the Gödel universe has to be described by a differential rotation law.

3. FIELD EQUATIONS FOR SHEARS

In paper [1] we have set up the field equations for the system (A). The evaluation of the field equations based on quantities deduced from the local non-rotating metric leads to a different structure. Rewriting the metric (B) once more with the physical quantities (2.9) we obtain

$$ds^2 = \mathcal{R}^2 d\eta^2 + dz^2 + \left[\alpha^{-1}\sigma d\varphi + i\alpha\omega\sigma\, dx^4\right]^2 + \alpha^2\, dx^{4^2} \ . \tag{3.1}$$

In non-relativistic approximation the metric reads as

$$ds^2 = \mathcal{R}^2 d\eta^2 + dz^2 + \left[\sigma\left(d\varphi - \omega dt\right)\right]^2 - dt^2 \ . \tag{3.2}$$

With the 4-bein vectors

$$\begin{array}{lllll} \overset{1}{e}_1 = 1, & \overset{2}{e}_2 = \dfrac{\sigma}{\alpha}, & \overset{3}{e}_3 = 1, & \overset{2}{e}_4 = i\alpha\omega\sigma, & \overset{4}{e}_4 = \alpha \\ \underset{1}{e}^1 = 1, & \underset{2}{e}^2 = \dfrac{\alpha}{\sigma}, & \underset{3}{e}^3 = 1, & \underset{4}{e}^2 = -i\alpha\omega\,, & \underset{4}{e}^4 = \dfrac{1}{\alpha} \end{array} \tag{3.3}$$

we can calculate the quantities

$$\begin{gathered} A_{2m}{}^{2} = \bar{B}_m = B_m - F_m - D_m, \quad A_{4m}{}^{4} = G_m = F_m + D_m \\ D_{mns} = 2\left[D_{(mn)}u_s - D_{(ms)}u_n + D_{[ns]}u_m\right] \ . \\ F_m = \alpha^2\omega^2\sigma\sigma_{|m}, \quad D_m = \alpha^2\omega\,\omega_{|m}\sigma^2 \end{gathered} \tag{3.4}$$

These quantities have only a few non-vanishing components

$$B_1 = \frac{1}{\mathcal{R}}\mathrm{cth}\eta, \quad F_1 + D_1 = \omega\,\mathrm{sh}\theta\,\mathrm{ch}\theta, \quad D_{12} = i\omega\left[\mathrm{sh}^2\theta - \mathrm{sh}^2\chi\right], \quad D_{21} = 0 \ . \tag{3.5}$$

$\bar{B}_m$ is the curvature quantity of the 3-dimensonal geometry, F_m the centripetal force and D_m a contribution from the differential rotation law. The D_{mn} are the shears. The Ricci-rotation coefficients are composed of all these quantities

$$\begin{gathered} A_{mn}{}^{s} = b_m\bar{B}_n b^s - b_m b_n \bar{B}^s + D_{mn}{}^{s} + u_m G_n u^s - u_m u_n G^s \\ b_m = \{0,1,0,0\}, \quad u_m = \{0,0,0,1\} \end{gathered} \ . \tag{3.6}$$

The u_m are the 4-velocities of the observers of the *locally non-rotating reference system* (LNR). With these connexion coefficients one has

$$u_{m\|n}u^n = -(F_m + D_m), \quad u_{[\alpha\|\beta]} = 0, \quad u_{(\alpha\|\beta)} = -2D_{(\alpha\beta)} \ . \tag{3.7}$$

The last two relations show that the LNR observers experience no Coriolis-like forces but shears.

Defining the graded covariant derivatives

$$\bar{B}_{n\underset{2}{\|}m} = \bar{B}_{n|m}, \quad G_{n\underset{3}{\|}m} = G_{n|m} - \bar{B}_{mn}{}^{s}G_s \ , \tag{3.8}$$

we can write the Ricci as

$$\begin{aligned} R_{mn} = &-\left[\bar{B}_{n\underset{2}{\|}m} + \bar{B}_n\bar{B}_m\right] - \left[G_{n\underset{3}{\|}m} + G_n G_m + 2D_n{}^{s}D_{ms}\right] \\ &- b_m b_n\left[\left(B^s{}_{\underset{2}{\|}s} + B^s B_s\right) - \left(G^s{}_{\underset{3}{\|}s} + G^s G_s + 2D^{sr}D_{sr}\right)\right] \\ &- u_m u_n\left[G^s{}_{\underset{3}{\|}s} + G^s G_s + 2D^{sr}D_{sr}\right] \\ &- 2u_{(m}\left[D^s{}_{n)\underset{3}{\|}s} + D^s{}_{n)}\bar{B}_s\right] \end{aligned} \ . \tag{3.9}$$

Gödel’s matter tensor is transformed with (2.6) to

$$T_{mn} = \mu_0\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\alpha^2\omega^2\sigma^2 & 0 & -i\alpha^2\omega\sigma \\ 0 & 0 & 0 & 0 \\ 0 & -i\alpha^2\omega\sigma & 0 & \alpha^2 \end{pmatrix} = \mu_0\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\mathrm{sh}^2\theta & 0 & -i\,\mathrm{sh}\theta\,\mathrm{ch}\theta \\ 0 & 0 & 0 & 0 \\ 0 & -i\,\mathrm{sh}\theta\,\mathrm{ch}\theta & 0 & \mathrm{ch}^2\theta \end{pmatrix} \ . \tag{3.10}$$

Gödel used the extended field equations with the cosmological constant

$$R_{mn} - \frac{1}{2} R g_{mn} - \lambda g_{mn} = -\kappa T_{mn} \ . \tag{3.11}$$

As $\lambda = -\frac{R}{2}$ one is left with

$$R_{mn} = -\kappa T_{mn} \ . \tag{3.12}$$

For the radial part of the field equations one obtains

$$G^s{}_{\underset{3}{||}s} = \kappa \mu_0 \alpha^2 - G^s G_s - 2 D^{sr} D_{sr} \tag{3.13}$$

and for the shears

$$D^s{}_{n\underset{3}{||}s} = -\kappa \mu_0 i \alpha^2 \omega \sigma b_n - D^s{}_n \bar{B}_s \ . \tag{3.14}$$

The radial field strengths are coupled to the matter density and to the field energy including the shears. The shears have the matter current and a field contribution as source. Finally, we add some references concerning Gödel's model [5 – 57].

4. CONCLUSIONS

We have shown that our assertion that an observer in the Gödel universe can experience shears is aided by the introduction of a locally non-rotating system. If the metric is reordered and disassembled in 4-bein fields in a proper way one obtains from the Ricci-rotations coefficients field strengths which are responsible for the shears of the observer fields. These shears are closely related to a differential rotation law and satisfy the field equations.

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