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GEOMETRY AND COSMOLOGY

Rainer Burghardt[*]

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Abstract: To describe cosmological models, we use geometric methods based on embeddings of class one. The models we study are positively curved and closed and are based on a pseudo-hypersphere.

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[*] e-mail: arg@aon.at, arg@arg.or.at, home page: http://arg.or.at/

1. INTRODUCTION

We suggest that a detailed geometrical revision of cosmological models will be advantageous in the understanding of their physical content. We treat exclusively positively curved models in this paper, because we believe that infinite worlds with infinite matter created at the Big Bang are not realized by Nature. We use embeddings of curved spaces in higher dimensional flat spaces as a geometric method and restrict ourselves to embeddings of class one. The 2^{nd} fundamental forms, the Gaussian equations, and the Codazzi equations also play an essential role.

2. THE PSEUDO-HYPERSPHERE

In our considerations, the geometric object, i.e., the pseudo-hypersphere, has a fundamental meaning. Therefore, we connect to historical methods of embedding geometric objects in a 5-dimensional fictitious space.

In a 5-dimensional flat space spanned by a Cartesian coordinate system $a' = 0', 1', \ldots 4'$, a family of pseudo-hyperspheres with radii $\mathcal{R}$ is defined by the equation

$$x^{a'} x^{a'} = \mathcal{R}^2 . \tag{2.1}$$

In pseudo-spherical coordinates

$$\begin{aligned} x^{3'} &= \mathcal{R} \sin\eta \sin\vartheta \sin\varphi \\ x^{2'} &= \mathcal{R} \sin\eta \sin\vartheta \cos\varphi \\ x^{1'} &= \mathcal{R} \sin\eta \cos\vartheta \qquad , \\ x^{4'} &= \mathcal{R} \cos\eta \sin i\psi \\ x^{0'} &= \mathcal{R} \cos\eta \cos i\psi \end{aligned} \tag{2.2}$$

the line element in the 5-dimensional space has the form

$$ds^2 = d\mathcal{R}^2 + \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \sin^2\eta d\vartheta^2 + \mathcal{R}^2 \sin^2\eta \sin^2\vartheta d\varphi^2 + \mathcal{R}^2 \cos^2\eta di\psi^2 . \tag{2.3}$$

If we select one of the pseudo-hyperspheres by means of the embedding condition

$$\mathcal{R} = \text{const.}, \tag{2.4}$$

we obtain one of these pseudo-hyperspheres and a 4-dimensional metric. The time interval is given by

$$\mathcal{R}\, di\psi = i\, dt \tag{2.5}$$

and represents the arc element on an (open) pseudo-circle. The metric has the index 4. We use the original Minkowski notation with $x^4 = i(c)t$.

From (2.3), we can read the components of the 5-beine, tangent to the local spherical coordinate system as:

$$\overset{0}{e}_0 = 1, \quad \overset{1}{e}_1 = \mathcal{R}, \quad \overset{2}{e}_2 = \mathcal{R}\sin\eta, \quad \overset{3}{e}_3 = \mathcal{R}\sin\eta\sin\vartheta, \quad \overset{4}{e}_4 = \mathcal{R}\cos\eta . \tag{2.6}$$

From these, we can immediately compute the reciprocal components. For the description of the problem, we exclusively use the pentad and the tetrad calculus. Consistently, for the partial derivatives in the tangent system of the pseudo-hypersphere with the local pseudo-spherical coordinates $a = 0,1,...4$, we have

$$\partial_0 = \frac{\partial}{\partial \mathcal{R}}, \quad \partial_1 = \frac{1}{\mathcal{R}}\frac{\partial}{\partial \eta}, \quad \partial_2 = \frac{1}{\mathcal{R}\sin\eta}\frac{\partial}{\partial \vartheta}, \quad \partial_3 = \frac{1}{\mathcal{R}\sin\eta\sin\vartheta}\frac{\partial}{\partial \varphi}, \quad \partial_4 = \frac{1}{\mathcal{R}\cos\eta}\frac{\partial}{\partial i\psi}. \quad (2.7)$$

For the radii of the family of pseudo-hyperspheres, the following applies:

$$\mathcal{R}_{|a} = n_a = \{1,0,...0\}. \quad (2.8)$$

Here, n_a is the normal vector of the pseudo-hypersphere, and n_0 is the component in the local extra dimension. In the following sections, we will allow a time dependence of $\mathcal{R}$ for expanding universes.

The Ricci-rotation coefficients [1] constitute the basis for a coordinate-invariant representation of a world model. We first compute their 0-components with (2.6) and (2.7). Consequently, only the quantities

$$A_{m0}{}^{s} = -\overset{s}{e}{}_{i}\, e^{i}{}_{m|0} = \frac{1}{\mathcal{R}}\delta^{s}_{m}, \quad A_{mn}{}^{0} = -\frac{1}{\mathcal{R}} g_{mn}. \quad (2.9)$$

remain. Here, $m = 1,2,...,4$ are 4-bein indices and $i = 1,2,...,4$ are coordinate indices.

For the covariant 5-dimensional derivative of the normal vector we find:

$$n_{m|||n} = n_{m|n} - A_{nm}{}^{s} n_s = -A_{nm}{}^{0}.$$

We recognize

$$A_{mn} = n_{m|||n}, \quad A_{[mn]} = 0, \quad A_{mn} = \frac{1}{\mathcal{R}} g_{mn} \quad (2.10)$$

as 2nd fundamental forms of the surface theory. They can be used to represent the essential properties of a cosmological model.

Next, we separate the 0-components from the Riemann and decompose the Ricci-rotations coefficients according to

$$A_{ab}{}^{c} = {}'A_{ab}{}^{c} + \Delta_{ab}{}^{c}. \quad (2.11)$$

'A now contains only 4-dimensional components, whose structure depends on the embedded surface, in our case, on the structure of the pseudo-hypersphere. The tetrads and coordinate system should be adapted to the geometrical structure as best as possible so that the calculations run smoothly.

We summarize the previously calculated fundamental forms in the quantity

$$\Delta_{ab}{}^{c} = A_{a}{}^{c} n_b - A_{ab} n^{c} \quad (2.12)$$

The 4-dimensional covariant derivative, which refers to the surface of the pseudo-hypersphere, is given by

$$\Phi_{m||n} = \Phi_{m|n} - {}'A_{nm}{}^{s}\Phi_s. \quad (2.13)$$

In particular, we have

$$n_{m||n} = 0. \quad (2.14)$$

The 5-dimensional Riemann in pentad form

$$R_{dab}{}^{c} = 2\left[A_{[a\cdot b\cdot}{}^{c}{}_{|||d]} - A_{[a\cdot b\cdot}{}^{g} A_{d]g}{}^{c}\right] \equiv 0 \quad (2.15)$$

is identically zero, since we assume a flat embedding space. If we decompose according to (2.11) and use the covariant 4-dimensional derivative, we obtain after some calculations

$$R_{dab}{}^{c} = {}'R_{dab}{}^{c} + 2n_{[d}\left[{}'A_{a]b}{}^{c}{}_{||g}\, n^{g} + A_{a]}{}^{g}\, {}'A_{gb}{}^{c}\right] + \Delta_{dab}{}^{c}\,. \tag{2.16}$$

Here,

$$'R_{dab}{}^{d} = 2\left[{}'A_{[\underline{a}\cdot\underline{b}\cdot}{}^{c}{}_{||\underline{d}]} - {}'A_{[\underline{a}\cdot\underline{b}\cdot}{}^{g}\, {}'A_{\underline{d}]g}{}^{c}\right] \tag{2.17}$$

is the 4-dimensional Riemannian, which no longer contains 0-components. The underline means that the index refers to the 4-dimensional embedded space. Furthermore, we have

$$\Delta_{dab}{}^{c} = 2\left[n_{b}A_{[a}{}^{c}{}_{||d]} - n_{b}A_{[d}{}^{g}\, n_{a]}\, A_{g}{}^{c} - A_{[a\cdot b\cdot||d]}\, n^{c} + A_{[d}{}^{g}\, n_{a]}A_{gb}n^{c} + A_{[a}{}^{c}A_{d]b}\right]. \tag{2.18}$$

Considering (2.9),

$$A_{[a}{}^{c}{}_{||d]} - A_{[d}{}^{g}\, n_{a]}\, A_{g}{}^{c} = 0, \quad A_{[a\cdot b\cdot||d]} - A_{[d}{}^{g}\, n_{a]}A_{gb} = 0\,, \tag{2.19}$$

where the first relation in the four-dimensional form is the Codazzi equation

$$A_{[m}{}^{s}{}_{||n]} = 0\,, \tag{2.20}$$

which is trivially satisfied since the radius of curvature of the pseudo-hypersphere is constant. Finally there remains only

$$\Delta_{dab}{}^{c} = 2\,A_{[a}{}^{c}A_{d]b}\,. \tag{2.21}$$

From the parenthesis in (2.17) we take

$$A_{ab}{}^{c}{}_{||g}\, n^{g} + A_{a}{}^{g}\, {}'A_{gb}{}^{c}\,.$$

It essentially contains expressions of the spherical frame of reference, which can be found in the Mathematical Appendix A. A short calculation using (2.10) shows that the expression vanishes as well. For the pseudo-hypersphere there remains only:

$$'R_{dab}{}^{d} = -\Delta_{dab}{}^{c} = -2\,A_{[a}{}^{c}A_{d]b} \tag{2.22}$$

If we also use (2.10), we get for the 4-dimensional form

$$'R_{rmns} = \frac{1}{\mathcal{R}^{2}}\left[g_{mn}g_{rs} - g_{ms}g_{rn}\right]. \tag{2.23}$$

These are the Gaussian curvature equations for the embedding of an M_4 in a flat M_5.

By contraction, the Ricci tensor and Ricci scalar can be derived from these relations. With equations (2.16) - (2.18), we have revealed the general structure of a closed cosmological model. Further, with (2.20) and (2.23), we have reduced the general relations to those of a pseudo-hypersphere.

3. THE DE SITTER COSMOS

In the year 1916, de Sitter [2-6] proposed a matter-free cosmological model that contains the cosmological constant in the field equations. This model has raised questions about the validity of Mach's principle. According to Mach, gravitational effect should be determined by the total mass of space. However, gravitational forces are present in the empty de Sitter cosmos. We evaded the discussion of this problem by bringing the expression that has the cosmological constant to the right side of Einstein's field equations and identifying it with the pressure and the mass density.

The dS cosmos is geometrically based on a pseudo-hypersphere, as described in section 2. If we have selected a pseudo-hypersphere from the family, then the metric on it is

$$ds^2 = \mathcal{R}^2 d\eta^2 + \mathcal{R}^2 \sin^2\eta d\vartheta^2 + \mathcal{R}^2 \sin^2\eta \sin^2\vartheta d\varphi^2 + \mathcal{R}^2 \cos^2\eta d i\psi^2 . \tag{3.1}$$

If we define a radial variable

$$r = \mathcal{R} \sin\eta , \tag{3.2}$$

we can also express the metric in the form

$$ds^2 = \frac{1}{\cos^2\eta} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \cos^2\eta dt^2$$

or

$$ds^2 = \frac{1}{1-\frac{r^2}{\mathcal{R}^2}} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \left(1-\frac{r^2}{\mathcal{R}^2}\right) dt^2 . \tag{3.3}$$

We have listed the Ricci-rotations coefficients related to (3.1) in the Mathematical Appendix A.

In the previous section, we presented the geometry of the pseudo-hypersphere in detail. We can now refer to these results. From (2.23), we calculate the Ricci and Einstein tensors:

$$R_{mn} = \frac{3}{\mathcal{R}^2} g_{mn}, \quad R = \frac{12}{\mathcal{R}^2}, \quad G_{mn} = -\frac{3}{\mathcal{R}^2} g_{mn} . \tag{3.4}$$

This results in

$$\kappa p = -\frac{3}{\mathcal{R}^2}, \quad \kappa\mu_0 = \frac{3}{\mathcal{R}^2}, \quad p + \mu_0 = 0 . \tag{3.5}$$

The first two field quantities given in Appendix (A.1), describe the spherical structure of the model. More interesting, however, is the quantity U, which is calculated from the lapse function of (3.1) and points in the radial directions in the cosmos. Particles forming a swarm are driven apart in all directions in free fall. This has raised the question of where these particles eventually end up. Eddington conjectured that the matter of the cosmos is swept together on a mass horizon.

Finally, it was supposed that it is not the swarm but the cosmos itself that expands. Lemaître [7] and Robertson [8] have established a transformation to a coordinate system that comoves with the drifting particles. In the Appendix A, we have given this transformation in a notation corresponding to the form of metric (3.1). This coordinate transformation can be associated with a Lorentz transformation. We have presented the somewhat tedious conversion in detail in [1].

The metric in these coordinates has the form

$$ds^2 = \mathcal{K}^2\left[dr'^2 + r'^2 d\vartheta^2 + r'^2 \sin^2\vartheta\, d\varphi^2\right] + \mathcal{R}^2 di\psi'^2, \quad \mathcal{K} = e^{\psi'}, \quad \psi' = \frac{t'}{\mathcal{R}}, \tag{3.6}$$

where the variables of the comoving system are primed. The metric is of the type $k = 0$, where k is called the curvature parameter. This should indicate whether a space is positively curved $(k = 1)$, flat $(k = 0)$, or negatively curved $(k = -1)$. A Lemaître transformation transforms the metric (3.1) of type $k = 1$ into the metric (3.6) of type $k = 0$. This does not mean that the curvature of the space is changed by the coordinate transformation, but that an observer comoving with the expansion of the particle swarm perceives the space as locally flat. This is similar to the way an observer does in a freely falling elevator in the Schwarzschild field. The variable t' in (3.6) is the cosmic time valid for all observers and thus also the proper time of all drifting observers. This, and also the shape of the metric indicate that the particles of the swarm are in free fall.

The coordinate transformation can be associated with a Lorentz transformation as shown in detail in [1]. However, the Lorentz transformation can be easily obtained by reading the relative velocity of the particles from the lapse function of (3.3). From (3.2), we have

$$v = \frac{r}{\mathcal{R}} = \sin\eta\,, \tag{3.7}$$

and thus, also the Lorentz factor $\alpha = 1\big/\sqrt{1 - r^2/\mathcal{R}^2}$. If an observer defines his position on the pseudo-hypersphere as a pole, then the velocity of the particles would be the velocity of light at his equator. Thus, the equator $(\eta = \pi/2)$ is the cosmic horizon.

The field quantities of the comoving system can be calculated directly with (3.6) by reading the 4-beine from (A.3) and using them to determine the Ricci-rotation coefficients (A.4). Alternatively, we can use the inhomogeneous transformation law of the Ricci-rotations coefficients (A.5). The spatial part of the two lateral field quantities takes on a flat form. The quantity U transforms accordingly to

$$U_m = \left\{-\frac{1}{\mathcal{R}}\tan\eta, 0, 0, 0\right\} \rightarrow {}'U_{m'} = \left\{0, 0, 0, -\frac{i}{\mathcal{R}}\right\} \tag{3.8}$$

and makes it clear that acts no gravity in the comoving system, i.e., that the drifting particles are in free fall. As stated above, the common global time is also valid for them.

In the comoving system, the forces manifest themselves as tidal forces:

$${}'U_{4'} = B_{4'} = C_{4'} = -\frac{i}{\mathcal{R}}. \tag{3.9}$$

They expand a volume element in all three directions. However, the spatial parts of the lateral field quantities necessarily appear as flat:

$$B_{\alpha'} = \left\{\frac{1}{r}, 0, 0,\right\}, \quad C_{\alpha'} = \left\{\frac{1}{r}, \frac{1}{r}\cot\vartheta, 0,\right\}. \tag{3.10}$$

This is a consequence of Einstein's elevator principle [9]. In an elevator which is in free fall, no gravitational forces act, therefore, the curvature of space is not perceivable.

Thus, the dS cosmos appears locally flat in the comoving system, but is still globally curved.

The mathematical structure of the tidal forces suggests a second application of surface theory. In Appendix B, we repeated the mathematical methods which we presented in Section 2, but lowered the dimension of the embedding by one degree. The embedding vector of a 3-surface represented by the particle swarm is time-like

$$'u_{m'} = \{0,0,0,1\}, \tag{3.11}$$

and the tidal forces are identified with the 2^{nd} fundamental forms of the surface theory[1]:

$$D_{mn} = -\frac{i}{\mathcal{R}} {}^*g_{mn}. \tag{3.12}$$

Here, ${}^*g_{mn}$ is the 3-dimensional part of the metric.

For all further considerations, we can rely on the formulae (B.5) and (B.6). We will see that these relations simplify their application to the dS cosmos.

However, we have to consider that in the comoving system, several quantities depend on the global time t'. We calculate this dependence using

$$\partial_{4'} = -i\alpha v \partial_1 = -i\alpha v \frac{\partial}{\mathcal{R}\partial\eta} \text{ and } \partial_{4'} = \frac{\partial}{\mathcal{R}\partial i\psi'}. \tag{3.13}$$

Thus, we can show that the second and fourth terms in (B.5) vanish. Also, the Codazzi equation

$$D_{[m\ \wedge s]}^{\ \ s} = 0 \tag{3.14}$$

is trivially satisfied. The 3-dimensional Ricci is also zero, since the space is locally flat. There remains only

$$R_{mn} = -D_{mn}D_s^{\ s} - u_m u_n D_{rs}D^{rs}, \quad R = -D_{rs}D^{rs} - D_n^{\ n}D_s^{\ s}. \tag{3.15}$$

From this equation, we get:

$$G_{mn} = -\frac{3}{\mathcal{R}^2} g_{mn}, \quad \kappa p = -\frac{3}{\mathcal{R}^2}, \quad \kappa\mu_0 = \frac{3}{\mathcal{R}^2}, \quad p + \mu_o = 0. \tag{3.16}$$

These are the same values as in the static case (3.5). In the dS cosmos, in spite of the particle motion, no matter currents appear, pressure and matter density remain constant despite the expansion. Mitra [10] has pointed out in a paper the existence of contradictions concerning the transformation from a comoving to a non-comoving reference frame. Grøn [11] calls the absence of matter currents the Mitra paradox.

The unphysical behavior of p and μ_0 follows directly from the possibility of proceeding from the non-comoving system to the comoving system with a Lorentz transformation. Since the Ricci and the Einstein tensors are Lorentz-invariant, it follows from (3.16) that

$$G_{m'n'} = L^{m\ n}_{m'n'} G_{mn} = L^{m\ n}_{m'n'}\left[-\frac{3}{\mathcal{R}^2} g_{mn}\right] = -\frac{3}{\mathcal{R}^2} g_{m'n'}$$

and therefore also the formal invariance of the stress-energy-momentum tensor.

[1] We temporarily omit the primes at the indices and the kernels, since we are only working with the comoving system.

We illuminated the dS cosmos mathematically, and to further elaborate the problem, we have given another formal way of describing the embedding of the pseudo-hypersphere in Appendix C. We added a 0-component to the field quantities and allowed us to show that the norm of these quantities describes the curvature of the normal and oblique cuts of the pseudo-hypersphere:

$$M = \frac{1}{\mathfrak{R}}, \quad B = \frac{1}{\mathfrak{R}\sin\eta}, \quad C = \frac{1}{\mathfrak{R}\sin\eta\sin\vartheta}, \quad U = \frac{1}{\mathfrak{R}\cos\eta}.$$

The curvature equations of these quantities are all of the type $\frac{\partial}{\partial r}\frac{1}{r} + \frac{1}{r^2} = 0$, and are subequations of Einstein's field equations. By performing a [0+4] decomposition and bringing the 0-terms to the right-hand side of Einstein's field equations, we obtain the well-known expressions for the stress-energy-momentum tensor of the dS cosmos.

Schrödinger [12] and several other authors have interpreted the dS cosmos that it is not a particle swarm but the cosmos itself that is expanding. Schrödinger assumes a pseudo-hypersphere but demonstrates the expansion problem on a hyperboloid of revolution. Like Schrödinger, Rindler [13] interprets the dS cosmos as a hyperboloid and shows that time-like cuts on the hyperboloid through arbitrary planes correspond to uniformly accelerated particles. The problem has also been addressed by Robertson. To interpret a pseudocircle in a pseudoreal representation as a hyperbola may be illustrative; however, taking this representation literally can lead to errors.

The Schrödinger treatment of the dS cosmos probably dates back to a 1919 paper by Weyl [14], which explains the redshift of drifting galaxies on orbits that are cuts on the hyperboloid. Weyl relates the Doppler effect of moving light sources to the Einstein effect – the redshift between points of different gravitational field strength.

The discussion about how to understand the de Sitter model in its two versions occupied a wide space at that time. Finally, the eminent mathematician Klein was addressed. His detailed answer ended the discussion. It is not known whether Klein's authority or the argumentative content of the paper was decisive. However, we cannot find any connection between the geometries of the hyperspheres or their space-time sections and Klein's statements, nor have we found any paper that refers to Klein's publication.

A detailed report of the discussions and the correspondence of some scientists to this topic can be found in the work of Röhle [15].

The dS model has intrinsic inconsistencies and does not represent a cosmological model that can describe Nature. Nevertheless, its mathematical structure is quite interesting and can play a role in the construction of a more sophisticated model.

4. THE SUBLUMINAL MODEL

Since Friedman [16,17] it is known that an expanding universe can be described by a metric whose spatial part describes the metric of a hypersphere whose radius is time-dependent. The general form of such a metric in comoving coordinates will be:

$$ds^2 = \mathcal{K}^2 \left[\frac{1}{1-k\frac{r'^2}{\mathcal{R}_0^2}} dr'^2 + r'^2 d\Omega^2 \right] - dt'^2 . \tag{4.1}$$

Here, $\mathcal{K}$ is the position-independent but time-dependent scale factor, and Ω is the solid angle. k is the curvature parameter, which can take the values of 1, 0, -1 according to the FRW classification. For $k = 1$, the spatial curvature of the cosmos is supposed to be positive, and the cosmos is closed. $k = 0$ is called a flat and open space, and $k = -1$ a negatively curved and open space. $\mathcal{R}_0$ is a constant that can be absorbed by r'. $\{r', t'\}$ are comoving coordinates. In particular t' is the cosmic time, which is the same for all observers. If the metric factor is $g_{4'4'} = 1$, then the coordinate time coincides with the proper time for comoving observers. We call the form (4.1) of the metric the *canonical form*.

The fact that the universe is expanding in free fall is only given as a side note in the relevant papers. The consequences for the mathematical structure of a cosmological model are not discussed. Lemaître showed that for free fall in the Schwarzschild field and also for the particle swarm of the dS cosmos, there is a coordinate transformation, associated with observers in free fall. In this process, the shape of the metric changes from type $k = 1$ to type $k = 0$. This means that a free-falling observer feels no gravitational forces and believes space to be flat.

We are therefore critical about the interpretation of the quantity k as a curvature parameter and prefer to call it the form parameter of the metric. According to Einstein's elevator principle, which is also valid for cosmology [9], the space is locally flat for a free falling observer, but nevertheless *globally curved*. Therefore, in the Eq. (4.1) we have to set $k = 0$. It remains:

$$ds^2 = \mathcal{K}^2 \left[dr'^2 + r'^2 d\Omega^2 \right] - dt'^2 . \tag{4.2}$$

Our further considerations will be based on this relation.

However, the metrics appearing in the literature under the name FRW models allow the values $k \neq 0$, which contradicts the cosmic elevator principle. We have called them hybrid models [18]. If we use a Lorentz transformation by means of the recession velocity of the galaxies and proceed with this to a non-comoving system, we will arrive at a chaos of formulae that can hardly be assigned any physical meaning.

In papers on cosmology, the case $k = 0$ is also interpreted in such a way that the space is globally flat and therefore infinite. The Hubble law suggests that galaxies could move with superluminal velocity. Thus, in this case, no information exchange between the galaxies would be possible and it would come to a galactic island formation. Superluminal velocities are admitted with the argument that the galaxies remain in their position, however, the space expands.

We are reserved towards this scenario, since we think that whether galaxies are moving away from each other because of forces acting on them or because space is expanding is irrelevant. The laws of special relativity must be valid in both cases. This inevitably leads to the assumption that the space must be positively curved and closed.

If an observer defines his position as a pole on the hypersphere, the equator will be his cosmic horizon. From this observer’s view, the receding galaxies would reach the velocity of light at this location, but only after an infinitely long time.

The Friedman model, from which many expanding models are derived, is physically unsatisfactory because it is pressureless. Since the cosmos with its galaxies is regarded approximately as a space that is uniformly filled with matter dust, the expectation from a realistic model is that the stress-energy-momentum tensor will contain pressure. But the pressure is manually inserted into the stress-energy-momentum tensor on the right-hand side of the field equations, and the left side of the field equations is observed for its output..

Since such a differential equation system is underdetermining, in the solution of the field equations, free quantities remain, especially in the Friedman equation that describes the expansion of space. In the FRW models, these are the quantities Ω and the deceleration parameter. The latter contains an acceleration term that indicates forces that contradict Einstein's elevator principle and cannot be derived from the lapse function of the metric.

The existence of free quantities opens a wide field for fitting astrophysical measurements into these models by juggling these quantities. These models are called standard models, but they are not exact solutions of Einstein's field equations.

The R_h = ct model of Melia [19] and our Subluminal Model [20] are exact solutions of Einstein's field equations. Pressure and mass density result from the geometry of the models. The two models start from different approaches but ultimately lead to the same results, as shown in [21]. Here, we outline the main properties of our Subluminal Model.

We assume that the space in which we live in is a 3-dimensional hypersphere that expands. Its radius $\mathcal{R}=\mathcal{R}(t)$ depends on time. The mathematical framework for this model is based on well-known methods of surface theory. A short presentation can be found in Appendix B. To better understand the geometry of the Subluminal Model, only Eq. (B.5) needs to be adapted to the features of the model.

It should be remembered that the line element of the model only describes the curvature of space in terms of Gauss but not its change. For the latter, a second system of differential equation is needed, i.e., the Bianchi identities. In contracted form, they lead to the conservation law of the model and provide the change of the quantity $\mathcal{R}$ in time. The radius of curvature of the hypersphere $\mathcal{R}$ is the only and also necessary free parameter of the model.

The quantities $\frac{1}{\mathcal{K}}\mathcal{K}_{|4'}$ are derived from the tetrads of the line element (4.2). If we integrate the scale factor $\mathcal{K}$ by means of $\mathcal{R}=\mathcal{K}\mathcal{R}_0,\ \mathcal{R}_0=\text{const.}$ into the space curvature, we must consider the expression $\frac{1}{\mathcal{R}}\mathcal{R}_{|4'}$. Let us anticipate the result of this term. From the conservation law we obtain relations which can be applied to the Friedman equations (with $c=1$)

$$\mathcal{R}^{\boldsymbol{\cdot}}=1, \quad \mathcal{R}^{\boldsymbol{\cdot\cdot}}=0. \tag{4.3}$$

They state that the universe is expanding linearly, i.e., the expansion is not accelerating. This contradicts the results of Perlmutter and Riess, who interpreted their astrophysical data with an FRW model.

However, recent measurements from the PLANCK project have yielded values that, despite manipulation of the Ω parameters, cannot be fitted into the possible range of FRW models. Melia, on the other hand, proposed in a paper [22] that the results of the PLANCK project can be made consistent with a linear model. In his article, Melia also noted that the FRW models contain 10 paradoxes and inconsistencies related to 27 different types of observations.

According to the present state of science one can exclude the FRW models as a possibility to describe Nature. This is not only true because of their formal inconsistencies, but also because of their lack of agreement with astrophysical data.

To convince ourselves of the correctness of the relations mentioned in [20], we need only to consult the appendix B. In it the surface theory is formulated with the help of the 2nd fundamental forms. We have for the 2nd fundamental forms of the expanding hypersphere[2]

$$D_{mn} = -\frac{i}{\mathcal{R}} {}^*g_{mn} \tag{4.4}$$

and only have to consider the time dependence of $\mathcal{R}$ in the relations of Appendix B.

With (4.3), the calculations are simple. In Appendix B, we have shown that the quantity $^*R$ in (B.5) vanishes. Using the tetrad derivatives given there and the constituents of the spatial components of the Ricci-rotations coefficients from (A.4), we can show that the first brackets in (B.5) also vanish. The Codazzi equation contained in (B.5)

$$D_{[\alpha}{}^{\beta}{}_{\wedge\beta]} = 0 \tag{4.5}$$

is trivially satisfied by (4.4). The Gaussian equation becomes

$$R_{\gamma\alpha\beta\delta} = \frac{1}{\mathcal{R}^2}\left[g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\delta}g_{\gamma\beta}\right] \neq 0 \tag{4.6}$$

and confirms that the space is globally curved. From (B.5) and (B.6) then only remains

$$\begin{aligned} R_{\alpha\beta} &= -D_{\alpha\beta|4} - D_{\alpha\beta}D_{\gamma}{}^{\gamma} \\ R_{44} &= -D_{\alpha}{}^{\alpha}{}_{|4} - D_{\alpha\beta}D^{\alpha\beta} \\ R &= -2D_{\alpha}{}^{\alpha}{}_{|4} - D_{\alpha\beta}D^{\alpha\beta} - D_{\alpha}{}^{\alpha}D_{\beta}{}^{\beta} \end{aligned} \quad . \tag{4.7}$$

Eliminating the terms with the time derivatives using the Friedman equation (4.3), we have for the Ricci and the Einstein tensors

$$\begin{aligned} &R_{\alpha\beta} = \frac{2}{\mathcal{R}^2}g_{\alpha\beta}, \quad R_{44} = 0, \quad R = \frac{6}{\mathcal{R}^2} \\ &G_{\alpha\beta} = -\frac{1}{\mathcal{R}^2}g_{\alpha\beta} = \kappa p g_{\alpha\beta}, \quad G_{44} = -\frac{3}{\mathcal{R}^2} = -\kappa\mu_0 \end{aligned} \tag{4.8}$$

and finally the fundamental relations for the subluminal model. In [21] we have shown that the fundamental relations for the dS cosmos can be derived from this with $\mathcal{R} = \text{const.}$.

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[2] Here, the primes used in (4.2), are omitted. The asterisks denote the space-like part of the metric.

5. CONCLUSIONS

We showed that a linearly expanding model, i.e., a model whose expansion is not accelerated, can be derived from elementary relations of surface theory. We used the embedding theory for an M_n in an M_{n+1}, developed by the Italian mathematicians, especially by Levi-Civita, and represented by means of the 2nd fundamental forms. First, we put this theory into a modern form using the tetrad method and then applied it to a hypersphere. For a constant radius of the hypersphere, the dS cosmos was obtained, and for a time-dependent radius, our Subluminal Model was obtained. This gave the Gaussian curvature equations, the Codazzi equations and, finally, the physical relations describing the pressure and matter density of the universe. Both the dS and our Subluminal Model are exact solutions of Einstein's field equations. The latter does not require any additional parameters to fit astrophysical data to the models.

6. MATHEMATICAL APPENDIX A

From the Ricci-rotations coefficients for a pseudo-hypersphere we obtain the field quantities

$$B_m = \left\{\frac{1}{\mathcal{R}}\cot\eta, 0,0,0\right\}, \quad C_m = \left\{\frac{1}{\mathcal{R}}\cot\eta, \frac{1}{\mathcal{R}\sin\eta}\cot\vartheta, 0,0\right\}, \quad U_m = \left\{-\frac{1}{\mathcal{R}}\tan\eta, 0,0,0\right\}. \quad (A.1)$$

Here, U is a quantity pointing in the radial direction (1-direction)[3].

The coordinate transformation of Lemaître has the form:

$$r = \mathcal{K} r', \quad \mathcal{K} = e^{\psi'}, \quad \psi' = \psi + \ln\cos\eta, \quad e^{\psi'} = e^{\psi}\cos\eta, \quad t' = \mathcal{R}\psi'. \quad (A.2)$$

The 4-beine of the comoving system are:

$$\overset{1'}{e}_{1'} = \mathcal{K}, \quad \overset{2'}{e}_{2'} = r, \quad \overset{3'}{e}_{3'} = r\sin\vartheta, \quad \overset{4'}{e}_{4'} = 1, \quad r = \mathcal{K} r' \quad (A3)$$

The field quantities in the comoving system

$$'U_{m'} = \left\{0,0,0,-\frac{i}{\mathcal{R}}\right\}, \quad B_{m'} = \left\{\frac{1}{r}, 0,0,-\frac{i}{\mathcal{R}}\right\}, \quad C_{m'} = \left\{\frac{1}{r}, \frac{1}{r}\cot\vartheta, 0, -\frac{i}{\mathcal{R}}\right\} \quad (A.4)$$

are calculated with

$$'A_{14}{}^{1} = -\overset{1}{e}_{1}\, e^{i}_{1|4} = \frac{1}{\mathcal{K}}\mathcal{K}_{|4} = \frac{1}{e^{\psi'}}\frac{\partial}{i\partial t'}e^{\psi'} = -i\frac{\partial}{\partial t'}\frac{t'}{\mathcal{R}} = -\frac{i}{\mathcal{R}}. \quad (A.5)$$

The inhomogeneous transformation law with L as matrix of the Lorentz transformation of the Ricci-rotations coefficients is:

$$'A_{m'n'}{}^{s'} = L^{m\,n\,s'}_{m'n's} A_{mn}{}^{s} + L_{m'n'}{}^{s'}, \quad L_{m'n'}{}^{s'} = L^{s'}_{s} L^{s}_{n'|m'}. \quad (A.6)$$

[3] $x^4 = it,\ U_1 = A_{41}{}^{4} = -A_{t1}{}^{t},\ A_{t1}{}^{t} = +\frac{1}{\mathcal{R}}\tan\eta$

7. MATHEMATICAL APPENDIX B

This appendix presents the mathematical structure of expanding surfaces using the 2^{nd} fundamental forms of surface theory. We assume that the models discussed, contain expanding 3-spheres, whose rigging vectors

$$u_m = \{0,0,0,1\}$$

are time-like and perpendicular to the 3-sphere, i.e., perpendicular to its tangents. We define a symmetric spatial quantity as follows:

$$D_{mn} = u_{m||n}, \quad D_{[mn]} = 0, \quad D_{mn}u^n = 0\,. \tag{B.1}$$

Obviously, this quantity is part of the Ricci-rotation coefficients A:

$$u_{m||n} = u_{m|n} - A_{nm}{}^{s}u_s = -A_{nm}{}^{4} = D_{mn}\,.$$

By separating this quantity from the Ricci-rotation coefficients, we obtain

$$A_{mn}{}^{s} = {}^{*}A_{mn}{}^{s} + D_{mn}{}^{s}, \quad D_{mn}{}^{s} = D_m{}^{s}u_n - D_{mn}u^s, \quad D_{m(ns)} = 0\,. \tag{B.2}$$

Here, the $*A$ are some spatial components of the Ricci-rotation coefficients. By performing this decomposition in the Riemann

$$R_{rmn}{}^{s} = 2\left[A_{[m\cdot n\cdot}{}^{s}{}_{|r]} + A_{[m\cdot n\cdot}{}^{t}A_{r]t}{}^{s} + A_{[mr]}{}^{t}A_{tn}{}^{s}\right], \tag{B.3}$$

we obtain

$$\begin{aligned} R_{rmn}{}^{s} &= *R_{rmn}{}^{s} + 2n_{[r}\left[*A_{m]n}{}^{s}{}_{|t}u^t + D_{m]}{}^{t}*A_{tn}{}^{s}\right] \\ &\quad +2\left[u_nD_{[m}{}^{s}{}_{\wedge r]} - D_{[m\cdot n\cdot\wedge r]}u^s + u_nu_{[r}D_{m]}{}^{t}D_t{}^{s} + u^su_{[m}D_{r]}{}^{t}D_{tn}\right] + 2D_{[m}{}^{s}D_{r]n} \end{aligned}\,. \tag{B.4}$$

The quantity $*R_{rmn}{}^{s}$ is written with $*A$ analogous to (B.3). Here, the derivative with respect to $*A$ is defined by

$$\Phi_{m\wedge n} = \Phi_{m|n} - *A_{nm}{}^{s}\Phi_s, \quad u_{m\wedge n} = 0\,.$$

By contracting the Riemann, we get the Ricci tensor

$$R_{mn} = *R_{mn} - u_m\left[*A_{sn}{}^{s}{}_{|t}u^t + D_s{}^{t}*A_{tn}{}^{s}\right] + 2u_nD_{[m}{}^{s}{}_{\wedge s]} - D_{mn\wedge s}u^s - D_{mn}D_s{}^{s} - u_mu_nD_{rs}D^{rs}\,. \tag{B.5}$$

By contracting again, we obtain the Ricci scalar

$$R = *R - 2D_n{}^{n}{}_{\wedge s}u^s - D_{rs}D^{rs} - D_n{}^{n}D_s{}^{s}\,. \tag{B.6}$$

When applied to the models discussed, these formulae are drastically simplified.

That the quantity

$$*R_{rmn}{}^{s} = 2\left[*A_{[m\cdot n\cdot}{}^{s}{}_{|\underline{r}]} + *A_{[m\cdot n\cdot}{}^{t}*A_{r]t}{}^{s} + *A_{[mr]}{}^{t}*A_{tn}{}^{s}\right] \tag{B.7}$$

vanishes is to be assumed, because it is built up from the two field quantities $*B_{m'}$ and $*C_{m'}$, which describe a locally flat geometry. We take the first three components from the expressions of the pseudo-hypersphere from (A.4). We recall the relation between noncomoving and comoving coordinates $r = \mathfrak{K}\,r'$ and differentiate using the tetrad calculus with $\partial_{1'} = \partial/\mathfrak{K}\partial r'$ and $\partial_{2'} = \partial/\mathfrak{K}r'\partial\vartheta$. Thus, we obtain the curvature equations of the two cuts on the hypersphere and thus, if we again omit the primes at the indices:

$$*R_{rmn}{}^{s} = 0, \quad *R_{mn} = 0, \quad *R = 0. \tag{B.8}$$

For the 3-dimensional part in (B.4) only the Gauss equation remains

$$R_{\gamma\alpha\beta\delta} = D_{\alpha\delta}D_{\gamma\beta} - D_{\alpha\beta}D_{\gamma\delta}, \tag{B.9}$$

which indicates that the 3-dimensional subspace of the model is positively curved.

8. MATHEMATICAL APPENDIX C

From (II.2.3), we read the components of the 5-bein

$$\overset{0}{e}_0 = 1, \quad \overset{1}{e}_1 = \mathfrak{R}, \quad \overset{2}{e}_2 = \mathfrak{R}\sin\eta, \quad \overset{3}{e}_3 = \mathfrak{R}\sin\eta\sin\vartheta, \quad \overset{4}{e}_4 = \mathfrak{R}\cos\eta \tag{C.1}$$

and can immediately calculate from it the reciprocal components. The lapse function $\overset{4}{e}_4$ is space-dependent and leads to the field strength U. The unit vectors in the local system are:

$$m_a = \{0,1,0,0,0\}, \quad b_a = \{0,0,1,0,0\}, \quad c_a = \{0,0,0,1,0\}, \quad u_a = \{0,0,0,0,1\}.$$

The Ricci-rotation coefficients split into:

$$\begin{gathered} A_{ab}{}^{c} = M_{ab}{}^{c} + B_{ab}{}^{c} + C_{ab}{}^{c} + U_{ab}{}^{c} \\ M_{ab}{}^{c} = m_a M_b m^c - m_a m_b M^c, \;\; B_{ab}{}^{c} = b_a B_b b^c - b_a b_b B^c. \\ C_{ab}{}^{c} = c_a C_b c^c - c_a c_b C^c, \;\; U_{ab}{}^{c} = u_a U_b u^c - u_a u_b U^c \end{gathered} \tag{C.2}$$

The field strengths therein have been derived from the curvatures of the pseudo-hyperspheres

$$\begin{gathered} M_a = \left\{\frac{1}{\mathfrak{R}},0,0,0,0\right\}, \quad B_a = \left\{\frac{1}{\mathfrak{R}},\frac{1}{\mathfrak{R}}\cot\eta,0,0,0\right\} \\ C_a = \left\{\frac{1}{\mathfrak{R}},\frac{1}{\mathfrak{R}}\cot\eta,\frac{1}{\mathfrak{R}\sin\eta}\cot\vartheta,0,0\right\}, \quad U_a = \left\{\frac{1}{\mathfrak{R}},-\frac{1}{\mathfrak{R}}\tan\eta,0,0,0\right\}. \end{gathered} \tag{C.3}$$

$\mathfrak{R}$, $\mathfrak{R}\sin\eta$, $\mathfrak{R}\sin\eta\sin\vartheta$, $\mathfrak{R}\cos\eta$ are the curvature radii of the normal and inclined slices of the pseudo-hyper surfaces. The 5-dimensional field equations

$$R_{ab} \equiv 0 \tag{C.4}$$

decouple to the individual curvature equations

$$
\begin{aligned}
&M_{\underset{1}{a|||b}} + M_a M_b = 0, \quad M^c{}_{\underset{1}{|||c}} + M^c M_c = 0 \\
&B_{\underset{2}{a|||b}} + B_a B_b = 0, \quad B^c{}_{\underset{2}{|||c}} + B^c B_c = 0 \\
&C_{\underset{3}{a|||b}} + C_a C_b = 0, \quad C^c{}_{\underset{3}{|||c}} + C^c C_c = 0 \\
&U_{\underset{4}{a|||b}} + U_a U_b = 0, \quad U^c{}_{\underset{4}{|||c}} + U^c U_c = 0
\end{aligned}
, \qquad \text{(C.5)}
$$

where the graded derivatives were defined by

$$
\begin{gathered}
m_{\underset{1}{a|||b}} = m_{a|b} = 0, \quad b_{\underset{2}{a|||b}} = b_{a|b} - M_{ba}{}^c b_c = 0 \\
c_{\underset{3}{a|||b}} = c_{a|b} - M_{ba}{}^c c_c - B_{ba}{}^c c_c = 0, \quad u_{\underset{4}{a|||b}} = u_{a|b} - M_{ba}{}^c u_c - B_{ba}{}^c u_c - C_{ba}{}^c u_c = 0
\end{gathered}
.
$$

In four dimensions, the curvature equations do not decouple any more. If one accomplishes a dimensional reduction by shifting all 0-components of the Ricci tensor to the right sides of the relations, we obtain with m = 1,2,3,4

$$
\begin{aligned}
R_{mn} = &-\left[B_{\underset{2}{n||m}} + B_n B_m\right] - b_n b_m \left[B^s{}_{\underset{2}{||s}} + B^s B_s\right] \\
&-\left[C_{\underset{3}{n||m}} + C_n C_m\right] - c_n c_m \left[C^s{}_{\underset{3}{||s}} + C^s C_s\right] \\
&-\left[U_{\underset{4}{n||m}} + U_n U_m\right] - u_n u_m \left[U^s{}_{\underset{4}{||s}} - U^s U_s\right] \\
= &\, m_n m_m \left[M_0 B_0 + M_0 C_0 + M_0 U_0\right] \\
&+ b_n b_m \left[B_0 M_0 + B_0 C_0 + B_0 U_0\right] \\
&+ c_n c_m \left[C_0 M_0 + C_0 B_0 + C_0 U_0\right] \\
&+ u_n u_m \left[U_0 M_0 + U_0 B_0 + U_0 C_0\right]
\end{aligned}
. \qquad \text{(C.6)}
$$

The right side of (C.6) can be significantly simplified with the help of (C.3).

The 0-components of the curvature quantities can be interpreted as 2nd fundamental forms of the surface theory

$$
M_0 = A_{11}, \quad B_0 = A_{22}, \quad C_0 = A_{33}, \quad U_0 = A_{44}.
$$

Doing so, we have adjusted the 5-dimensional notation to the methods of the 2nd fundamental forms.

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