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THE INTERIOR OF AN ELLIPTIC STELLAR OBJECT

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Abstract: An elliptic-hyperbolic static model is established – a model that can be applied as a source for a gravitational field. Field equations as curvature equations of the geometry are presented. We find relations to the Kerr geometry.

Contents

1. INTRODUCTION

We present a gravitational model for a stellar object based on an oblate ellipsoid of revolution. The properties are closely related to the properties of the Kerr geometry. Instead of starting with a metric and specializing the coefficients to provide the desired model, we face a surface and calculate the radii of the curvature of the normal and oblique slices on that surface. With these we calculate the curvature equations of type $\frac{\partial}{\partial r}\frac{1}{r}+\frac{1}{r^2}=0$ and combine them to form the Ricci. From the Einstein tensor, we derive the stress-energy-momentum tensor and check whether its divergence vanishes.

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Equally, if a model can be geometrized, i.e., represented by a surface, the Ricci can be decomposed into subequations that describe the curvatures of the surface. This provides a deeper understanding of the model. For the exterior Kerr theory, we described this methodology in detail in one of our papers [2] and supported it with drawings.

To be able to use this procedure, it is necessary to carry out the theory in tetrad calculus. As a consequence of this methodology the above-mentioned curvature quantities turn out to be components of the Ricci-rotation coefficients. In addition, the original Minkowski notation $x^4 = i(c)t$ must be retained.

2. THE ELLIPTIC-HYPERBOLIC ANSATZ

Since we want to describe the interior of an elliptical stellar object, we choose an oblate ellipsoid of revolution as the basic element of the theory with the metric

$$ds^2 = \rho_H^2 \, d\theta_H^2 + \rho_E^2 \, d\theta_E^2 + \rho_C^2 \, d\theta_C^2 . \tag{2.1}$$

The families of ellipses that parametrize the object are confocal and have the eccentricity $a = \text{const.}$. The orthogonal trajectories of the ellipses are hyperbolae. They fix the 'radial' direction of the model. Here, ρ_H are the radii of curvature of the hyperbolae, ρ_E are the radii of curvature of the ellipses, and ρ_C are the radii of curvature of the circular sections of the ellipsoid of revolution. θ_H and θ_E are the ascent angles of the curvature vectors, and θ_C the angles at the circular slices. As the radii ρ_H and ρ_E are orthogonal, one has $d\theta \stackrel{d}{=} d\theta_E = -d\theta_H$.

Using the formulae and definitions (A1) and (A2) in the appendix, we record the metric in the usual Boyer-Lindquist coordinates:

$$ds^2 = a_R^2 dr^2 + \Lambda^2 d\vartheta^2 + \sigma^2 d\varphi^2 , \tag{2.2}$$

an expression familiar with the Kerr metric. Then we curve the space by introducing the following quantities

$$\cos\eta = \sqrt{1 - r^2/\mathcal{R}^2}, \quad \sin\eta = \frac{r}{\mathcal{R}}, \quad r < \mathcal{R} . \tag{2.3}$$

We try an embedding into a 4-dimensional flat space of class one with the Cartesian coordinates $x^{a'}$

$$\begin{aligned} x^{0'} &= \pm\sqrt{\mathcal{R}^2 - r^2} \\ x^{1'} &= r\cos\vartheta \\ x^{2'} &= A\sin\vartheta\cos\varphi \\ x^{3'} &= A\sin\vartheta\sin\varphi \end{aligned} . \tag{2.4}$$

Suppressing the third dimension and choosing the negative sign in (2.4) gives the following surface:

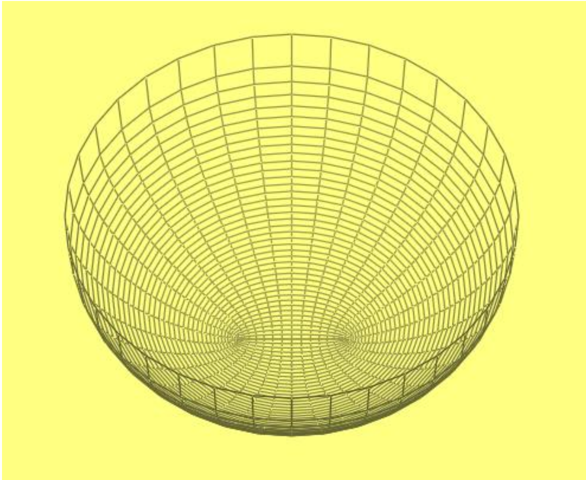

Fig. 1. The elliptic-hyperbolic surface

$\mathcal{R}$ is the radius of the circular arc at the minor semi-axes of the ellipses. All individual 'radial' curves have hyperbolic contributions in their properties. For r = 0, A = a, the horizontal ellipses are reduced to a distance clamped in by the common foci of the ellipses. If one now adds the third dimension, these points rotate through φ. For $\vartheta = \pi/2$ there emerges a circle. The radius of curvature of the ellipses is zero on this circle, and the assigned field quantities are infinitely large. This is the Kerr ring singularity.

Restricting ourselves to the first two dimensions of (2.4) and to the minor semi-axes of the ellipses we have

$$dx^{0'} = -\frac{r}{\mathcal{R}} \frac{1}{\sqrt{1-\frac{r^2}{\mathcal{R}^2}}} dr = -\tan\eta dr, \quad dx^{1'} = dr$$

and

$$dx^{0'2} + dx^{1'2} = \frac{1}{1-\frac{r^2}{\mathcal{R}^2}} = \frac{1}{\cos^2\eta} dr^2$$

Thus, we obtain a surface with a positive curvature parameter $k = 1$. Writing down the full line element we recognize that it does not have the desired structure.

Here we must bear in mind that dr is the increase of the minor axes of the ellipses on the base plane of the surface. We define the quantity

$$dx^{\underline{1}} = a_R dr \tag{2.5}$$

depending on the angle ϑ. This quantity is the distance between two infinitesimal neighboring ellipses. If one pulls up elliptical cylinders on two such ellipses in the base plane, one can see that their cutting curves with the surface do not lie on the horizontal slices of the surface. During circulation on the surface, the points of the cutting curve oscillate.

For our model only the horizontal elliptical slices of the surface are important. If one follows the normal vector of the surface along an ellipse, one will discover that this vector also oscillates along its way, because the walls of the surface are round about differently precipitous. To be able to use the surface, it must be equipped with an additional structure. On the minor axes of the ellipses the elliptical factor is $a_R = 1$, and the geometry is Schwarzschild-like. We make a start, and we define a *rigging vector* in such a way that it coincides with the normal vector at this position and that it always encloses the same angle with the base plane during circulation. Then, this rigging vector is no longer vertical to the surface and its vertical planes are no longer tangent to the surface. The family of all these planes – and if one adds the φ-dimension, the family of the 3-dimensional hyperplanes – represents the graphic space of the model. These hyperplanes are anholonomic and are the local spaces for the geometrical quantities derived from the metric. A similar problem occurs with the exterior Kerr metric. In [1] we described the problem in more detail and supported it with a drawing. Finally, we enhance the geometry with a time-like dimension and end up with a metric

$$ds^2 = \alpha_I^2 a_R^2 dr^2 + \Lambda^2 d\vartheta^2 + \sigma^2 d\varphi^2 - a_T^2 dt^2, \quad \alpha_I = 1/\cos\eta \tag{2.6}$$

which will serve as a metric for the model, we want to investigate. a_T is the lapse function of the new metric and will be discussed later on.

3. THE FIELD QUANTITIES

As our model can be embedded into a 5-dimensional flat space, it is convenient to note the 5-dimensional components of the field quantities (a = 0,1,…4). Here, x^0 is the *local* extra dimension. The elliptic, hyperbolic, and circular curvature quantities are as follows:

$$\begin{aligned} B_a &= \left\{\frac{1}{\rho_E}\sin\eta, \frac{1}{\rho_E}\cos\eta, 0,0,0\right\}, \quad N_a = \left\{0,0,\frac{1}{\rho_H},0,0\right\} \\ C_a &= \left\{\frac{1}{\sigma}\sin\theta\sin\eta, \frac{1}{\sigma}\sin\theta\cos\eta, \frac{1}{\sigma}\cos\theta, 0,0\right\} = \frac{1}{\sigma}\sigma_{|a} \end{aligned} . \tag{3.1}$$

The curvature radii are noted in (A3). As a consequence of curving space an additional field quantity appears, its only component pointing into the local 0-direction

$$M_a = \left\{\frac{1}{\rho_I}, 0,0,0,0\right\}, \quad \rho_I = \mathcal{R} a_R . \tag{3.2}$$

Here, $\mathcal{R} = \text{const.}$ is the curvature radius at the semi-axis of the ellipses, where the elliptical factor $a_R = 1$. For $a_R = 1$, everywhere the surface would be the cap of a sphere with radius $\mathcal{R}$ and would represent the interior Schwarzschild solution.

It is evident that the norms of all these field quantities are reciprocal to the curvature radii. The field quantities are components of the Ricci-rotation coefficients. For dimensional reduction, it is sufficient to omit the first components in the brackets. In the appendix, we note some relations that we need to solve the field equation.

Having explained the curved elliptic-hyperbolic system and its embedding into a flat 5-dimensional space we specify the lapse function

$$a_T = \frac{1}{2}\left[\left(1 + 2\Phi_g^2\right)\cos\eta_g - \cos\eta\right]\Phi_g^{-2}, \quad \Phi_g^2 = \frac{r_g^2 + a^2}{r_g^2 - a^2} = \text{const.}. \tag{3.3}$$

Here, r_g and η_g are constants. Evidently, for $a = 0$ one has $\Phi_g^2 = 1$, and a_T is reduced to the lapse function of the interior Schwarzschild metric. As usual, we derive the force of gravity from the lapse function. If we move on the base plane in the direction (2.5) and calculate $a_{T|\underline{1}}$, we get a quantity normal to the base plane

$$E_{\underline{1}} = -\frac{1}{a_T} a_{T|\underline{1}} = -\frac{1}{\rho_G a_T}, \quad \rho_G = 2\mathfrak{R} a_R \Phi_g^2,$$

which we project in the local 0- and 1-directions:

$$E_c = \left\{ \frac{1}{\rho_G a_T} \cos\eta, -\frac{1}{\rho_G a_T} \sin\eta, 0, 0, 0 \right\}. \tag{3.4}$$

The component E_1 is pointing inwards, i.e., it is attractive. Consequently, for $a = 0$ this quantity is reduced to the force of gravity of the interior Schwarzschild solution.

Now, we have all field quantities at hand, and we need to calculate the Ricci and the Einstein. If we apply the graded derivatives [1] to the field quantities, we obtain curvature equations that can be composed to the 5-dimensional Ricci, which evidently has to vanish. Separating all 0-components and shifting them to the right of Einstein's field equations, we obtain the 4-dimensional stress-energy-momentum tensor of this model. We have outsourced this procedure to a paging file that can be downloaded by the interested reader from. http://arg.or.at/PendingPapers/HiElp.pdf.

This model can probably serve as a source for an elliptic galaxy. Here, the problem arises that a non-spherical region should be embedded in a spherical environment. Many unsatisfactory attempts have been made to solve this problem. Recently, a promising ansatz was made by Huber [3] to match two regions with different symmetries by deforming the adjacent boundaries of these regions. This might apply to this cosmological problem.

The model developed thus far may provide a static seed metric for a Kerr interior. As all ellipses, hyperbolae, and circles defined by (2.4) are the same as for the Kerr exterior surface we are able to combine these surfaces. From our new surface a band has to be cut off and the remaining surface has to be matched horizontally to the surface of the exterior Kerr metric. Thus, one obtains a surface for the complete Kerr seed metric. We demonstrate this in Fig. 2 and Fig. 3.

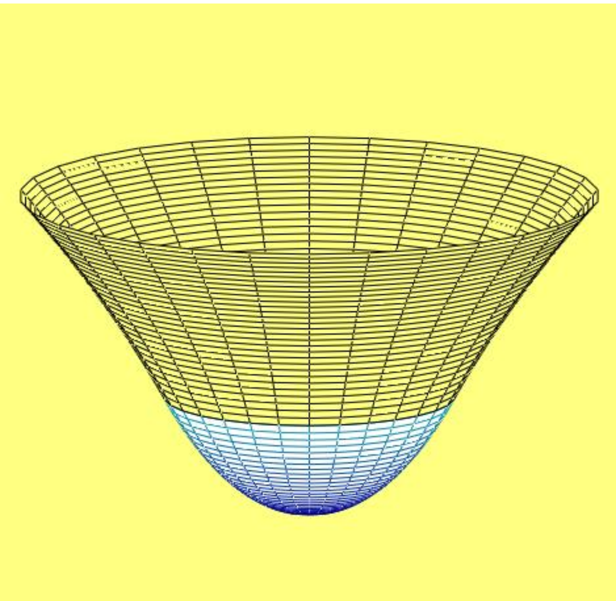

Fig. 3. Complete Kerr surface

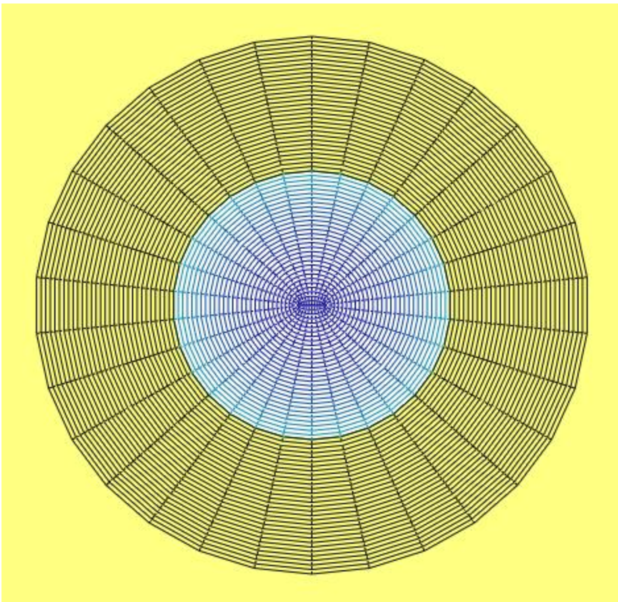

Fig. 4. Ground plane

We also admit a look at the center of the surface:

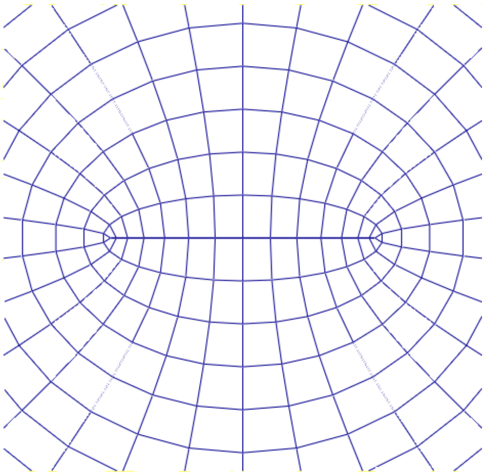

Fig. 5. Center of the interior surface

Now we have to check the linking conditions. It is evident that the hyperbolae and circles are continuous from the interior solution to the exterior solution. However, the 'radial' lines not only have a hyperbolic curvature, but also a 2nd curvature due to the curvature of space. There are numerous approaches in the literature for calculating the 2nd linking condition, which, however, prove to be less applicable for models that can be geometrized, i.e., they can be represented graphically by surfaces. We have shown in [4] that there is a simple solution for these models: at the boundary, the trestle lines of the two surfaces must have the same tangents (cutting tangents).

According to (2.4) we write for the extra dimension $R = x^{0'}$, we differentiate with respect to r, and we obtain

$$\frac{dR}{dr} = \pm \frac{r}{R}.$$

Since $R = \pm \mathcal{R}\sqrt{1 - r^2/\mathcal{R}^2} = \mathcal{R}\cos\eta$ and $r = \mathcal{R}\sin\eta$, we finally get

$$\frac{dR}{dr} = -\tan\eta, \quad \tan\eta = \frac{1}{\sqrt{\frac{\mathcal{R}^2}{r^2} - 1}}$$

for the lower part of the surface (2.4).

For the exterior Kerr solution [1] we found the ascent of the Kerr surface to be

$$\tan\varepsilon = -\sqrt{\frac{2Mr}{A^2 - 2Mr}}, \quad A^2 = r^2 + a^2,$$

with M as the mass of the field-generating source. If the ascents of the surfaces should coincide at the boundary $r = r_g$, it follows from $\tan\varepsilon_g = \tan\eta_g$

$$\mathcal{R} = A_g \sqrt{\frac{r_g}{2M}}. \tag{3.5}$$

This relation fixes the size of the cap matched to the exterior surface. For $a = 0$, one gets

$$2\mathfrak{R}=\sqrt{\frac{2r_g^3}{M}}=\rho_g\,,$$

where ρ_g is the curvature radius of Flamm's paraboloid at the boundary and $\rho_g=2\mathfrak{R}$ is Flamm's equation found in 1916.

Finally, we have to investigate the linking condition of the time-like part of the model. We write the time-like arc element as

$$dx^4=a_T idt=a_T\rho_g di\psi\,.$$

Here, $\rho_g=2\mathfrak{R}\Phi_g^2=\text{const.}$ is the curvature radius of the radial lines of the exterior Kerr surface [1] at the boundary. Thus, one obtains with (3.3)

$$dx^4=\left[\left(1+2\Phi_g^2\right)\mathfrak{R}\cos\eta_g-\mathfrak{R}\cos\eta\right]di\psi\,,$$

that is the arc element of two (open) pseudo circles. At the boundary surface, one has

$$dx^4=2\mathfrak{R}\Phi_g^2\cos\eta_g di\psi=\rho_g\cos\eta_g di\psi$$

which coincides with the corresponding expression for the boundary circle of the exterior surface. Thus, the 1st linking condition is satisfied. As the two pseudo circles coincide, they have trivially the same tangent and the 2nd liking condition is also satisfied.

In the paging file we have calculated the stress-energy-momentum tensor of the model. The pressure is anisotropic due to the axial symmetry of the source. We have

$$\begin{aligned}\kappa p_1&=-C_0B_0+C_0E_0+B_0E_0\\ \kappa p_2&=-2M_0C_0+C_0B_0+M_0E_0-B_0E_0+2C_0E_0\\ \kappa p_3&=\;\;2B_0B_0-2M_0B_0-C_0B_0+M_0E_0+C_0E_0\\ \kappa\mu_0&=-2B_0B_0+2M_0C_0+2M_0B_0+C_0B_0\end{aligned}\,. \tag{3.6}$$

For $a=0$, one obtains the values of the Schwarzschild interior solution. However, the greatest mathematical effort was the proof of the conservation law for these quantities. We performed the calculations in the paging file, and proceeded step by step. We first treated a simplified model, brought it into final form, and verified the conservation law.

Evidently, all three pressures contain the quantities E_0 of (3.6) having the lapse function (3.3) in the denominator. In the center of the source one has $r=0,\ \eta=0,\ \cos\eta=1$. For $a_T=0$, the quantity E_0 blows up and the pressures would be infinite. Thus, for the surface of the object with $r=r_g$, there is a limit r_g^{min} for its extent given by

$$\cos\eta_g=\frac{1}{1+2\Phi_g^2},\quad \cos^2\eta_g=1-\frac{r_g^2}{\mathfrak{R}^2}\,. \tag{3.7}$$

As a consequence, the stellar object cannot shrink beyond r_g^{min} and a continuous contraction to the center is impossible. The same should hold for a rotating version. We expect that the final state of a collapsing Kerr interior cannot be a black hole but rather a RECO (Rotating Eternally Collapsing Object).

From (3.7), we obtain for $a=0$ the relation $\cos\eta_g=1/3$, which leads to the well-known Schwarzschild limiting value $r_g^{min}=2.25\,M$. We [5] showed that the collapse of the

Schwarzschild interior indeed leads to an ECO (Eternally Collapsing Object). Mitra [6] covered this problem in full in his textbook.

The question is how to extend this model to a rotating model and match it to the exterior Kerr solution. In an earlier paper [7] we presented a trial solution. In doing so, we extended the differential rotation law of the exterior metric to the interior, which caused a high-velocity problem. Certainly, Nature does not support such behavior of a stellar object. The required task would be to find a better ansatz for a rotating model.

Rotation is implemented into the Kerr model by an anholonomic transformation of the coordinates. In general, for anholonomic geometry, Bianchi identities differ from the Riemannian structure. They exhibit additional terms, and their contractions do not lead to the conservation law of the model. We [8] studied this anholonomy problem and showed that for a rotation into the φ-direction, the Bianchi identities take the common Riemannian structure, although they contain anholonomic contributions. This has no meaning for the exterior solution because it is a vacuum solution, and the stress-energy-momentum tensor vanishes. In contrast, for the interior solution we have to convince ourselves that a conservation law exists despite anholonomy.

To construct a rotating Kerr interior several conditions must be fulfilled. The interior seed metric has to match the exterior seed metric, and the 1^{st} and 2^{nd} linking conditions have to be satisfied. The radial pressure at the boundary surface has to vanish to guarantee a stable object. Further, the anholonomy properties of the rotational effects should be well behaved.

4. CONCLUSIONS

We have proposed a curved elliptic-hyperbolic model that is a candidate for the seed metric for a Kerr interior. The model satisfies both the 1^{st} and 2^{nd} linking condition. it can be geometrized, i.e., represented graphically by a surface matching the exterior Kerr surface. The main problem was to set up the field equations and to prove the conservation laws. As this requires a remarkable amount of algebra, we decided to outsource these calculations in a paging file. We hope that we have given a functional starting model for a genuine Kerr interior.

5. APPENDIX

The transition to the Boyer-Lindquist coordinates can be established with

$$\sin\theta = \frac{r}{\Lambda}\sin\vartheta, \quad \cos\theta = \frac{A}{\Lambda}\cos\vartheta, \quad \rho_C = \sigma, \quad \theta_C = \varphi . \tag{A.1}$$

Here, r is the measure on the semi-minor axes, A on the semi-major axes of the ellipsoids, Λ is the geometrical mean of the focal rays and σ is the radius of the circular sections with the correlated angle φ. a_R denotes the elliptical factor. Details can be found in [1]. We note

$$A^2 = r^2 + a^2, \quad \Lambda^2 = r^2 + a^2\cos^2\vartheta, \quad a_R = \Lambda/A . \tag{A.2}$$

The curvature radii in Boyer-Lindquist coordinates are

$$\rho_E = \frac{\Lambda^3}{rA}, \quad \rho_H = -\frac{\Lambda^3}{a^2 \sin\vartheta\cos\vartheta}, \quad \rho_C = \sigma = A\sin\vartheta, \quad \rho_I = \Re a_R \,. \tag{A.3}$$

We note some derivatives of the basic quantities

$$\begin{aligned}
& r_{|a} = \{\sin\eta, \cos\eta, 0,0,0\}\,\alpha_R, \quad r_{|1} = a_I\alpha_R, \\
& \frac{1}{\Lambda}\Lambda_{|a} = \left\{\frac{1}{\rho_E}\sin\eta, \frac{1}{\rho_E}\cos\eta, \frac{1}{\rho_H}, 0,0\right\} \stackrel{*}{=} \{B_0, B_1, N_2, 0,0\} \\
& A_{|a} = \{\sin\eta, \cos\eta, 0,0,0\}\frac{r}{\Lambda}, \quad \frac{1}{A}A_{|a} \stackrel{*}{=} \{C_0, C_1, 0,0,0\} \quad . \\
& \sigma_a = \sigma_{|a} = \{\sin\theta\sin\eta, \sin\theta\cos\eta, \cos\theta, 0,0\}, \quad \sigma^c\sigma_c = 1 \\
& \theta_{|a} = \left\{-\frac{1}{\rho_H}\sin\eta, -\frac{1}{\rho_H}\cos\eta, \frac{1}{\rho_E}, 0,0\right\}
\end{aligned} \tag{A4}$$

For evaluating the field quantities we need the following relations

$$\begin{aligned}
& \rho_{E|0} = \left(1+\rho_E^2\Sigma^2\right)\sin\eta, \quad \rho_{E|1} = \left(1+\rho_E^2\Sigma^2\right)\cos\eta \\
& \rho_{E|2} = 3\frac{\rho_E}{\rho_H}, \quad \rho_{H|\underline{1}} = 3\frac{\rho_H}{\rho_E} \quad . \\
& \Sigma^2 = \frac{a^2}{\Lambda^4}\left(\sin^2\theta - \cos^2\theta\right)
\end{aligned} \tag{A5}$$

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