Harmonic and Wave Maps Coupled with Einstein's Gravitation

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Abstract

In this paper we discuss the coupled dynamics, following from a suitable Lagrangian, of a harmonic or wave map ϕ and Einstein's gravitation described by a metric g. The main results concern energy conditions for wave maps, harmonic maps from warped product manifolds, and wave maps from wave-like Lorentzian manifolds.

Keywords: Harmonic map; Wave map; Energy conditions.

1 Introduction

Scalar fields on a space or space-time manifold (X, g), which satisfy a linear or nonlinear field equation, attract enduring attention; cf, e.g., [1, 2, 3]. The *Dirichlet Lagrangian* or *energy density*

$$e = \frac{1}{2}|d\phi|^2 = \frac{1}{2}g^{ab}(\partial_a\phi)(\partial_b\phi), \tag{1.1}$$

of a one-component scalar field $\phi = \phi(x)$ leads to a linear field equation of Laplace or D'Alembert type. Here we denote

$$g = g_{ab}dx^a dx^b, \quad (g^{ab}) := (g_{ab})^{-1}, \quad \partial_a := \frac{\partial}{\partial x^a}.$$

The Dirichlet Lagrangian e admits a natural generalization to a multi-component scalar field $\phi = \phi(x)$ if the range of ϕ is suitably geometrized,

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namely if $im\,\phi$ lies in a Riemannian manifold (Y, h). That means, ϕ becomes a map $X \to Y$, $x \mapsto y$ between a source manifold (X, g) and a target manifold (Y, h). The choice of the Lagrangian

$$e = e[\phi] = e[\phi, g] = \frac{1}{2}|d\phi|^2, \tag{1.2}$$

where now

$$|d\phi|^2 := tr(\phi^* h) \equiv g^{ab}(\partial_a \phi^i)(\partial_b \phi^j) h_{ij}(\phi^k),$$

leads to the generally semi-linear field equation with Laplace-like or D'Alembert-like principal part

$$tr(\nabla d\phi)^i \equiv g^{ab} \nabla_a \partial_b \phi^i = 0. \tag{1.3}$$

Here we denote $h = h_{ij} dy^i dy^j$ and the special covariant derivative ∇ is built from g, h, ϕ as will be explained in section 2.

A map $\phi : X \longrightarrow Y$ between properly Riemannian manifolds (X, g), (Y, h) which satisfies (1.3) is called a *harmonic map* [4, 5]. A map $\phi : X \longrightarrow Y$ from a Lorentzian manifold (X, g) to a properly Riemannian manifold (Y, h) which satisfies (1.3) is called a *wave map* [6, 7, 8, 9, 3].

In this paper, we take (Y, h) as a fixed background and consider g and ϕ as dynamical objects. The dynamics shall follow from the Lagrangian

$$L = \kappa R - e, \tag{1.4}$$

where R denotes the scalar curvature of g and $\kappa \neq 0$ is a coupling constant. Variation with respect to g yields, for $dim X \geq 3$, the *Einstein equation* in the form

$$\kappa Ric = \phi^* h, \tag{1.5}$$

in components

$$\kappa R_{ab} = (\partial_a \phi^i)(\partial_b \phi^j) h_{ij}. \tag{1.6}$$

Variation with respect to ϕ yields (1.3).

A Lorentzian metric g can be interpreted as *gravitation*. According to the Kaluza-Klein principle, the space-time manifold (X, g) may have a higher dimension.

A positive definite metric g on X can be given a physical interpretation through a Lorentzian metric constructed from it as follows. Consider the product manifold $\tilde{X} := \Re \times X$ with points (t, x) and equip it with the Lorentzian metric $\tilde{g} := dt^2 - g$. Extend $\phi : X \longrightarrow Y$ to $\tilde{\phi} : \tilde{X} \longrightarrow Y$ by setting $\tilde{\phi}(t, x) := \phi(x)$, that means $\tilde{\phi} := \phi \circ pr_2$, where pr_2 is the projection

$\tilde{X} \longrightarrow X$ to the second factor. The field equations (1.5), (1.3) for $(\tilde{X}, \tilde{g})$, $\tilde{\phi}$ reduce to (1.5), (1.3) for (X, g), ϕ.

Let us sketch our main results.

• The energy-momentum tensor T of a wave map obeys several energy conditions. In particular, if v is a causal (i.e. non-spacelike) vector field then the momentum one-form $I := T(\cdot, v)$ is causal again.

• The Einstein equation (1.5) implies that the conditions $Ric(v, v) = 0$, $Ric(\cdot, v) = 0$, $\phi_* v = 0$ for a vector field v are equivalent to each other. Moreover, then v is a Ricci collineation, i.e. $\pounds_v Ric = 0$. Some conclusions are drawn from this latter fact.

• A submersive map $\phi : X \to Y$ between a pure manifold X and a Riemannian manifold (Y, h) can locally be made to a harmonic or wave map by a suitable choice of a metric g on X.

• We study the case of warped product $X = {}'X \times {}''X$, $g = {}'g \oplus w^2\, {}''g$. Several propositions are proved by means of the argument that the integral of a Laplace expression over a closed manifold ${}'X$ vanishes.

• We study radiation conditions for a Lorentzian manifold (X, g). The Einstein equation (1.5) leads from one condition to a stronger condition.

2 Preliminaries

We consider dimensions

$$m := dimX \geq 3, \quad n := dimY \geq 1$$

and adopt the following index convention: indices $a, b, c, ...$ label the components of geometric objects on X; indices $i, j, k, ...$ label the components of geometric objects on Y.

Tensor fields on Y become multi-component scalar fields on X by insertion of $y = \phi(x)$, where $x \in X$, $y \in Y$. For covariant tensor fields on Y there is also the conventional pull-back map ϕ^*. For instance, the metric $h = h_{ij} dy^i dy^j$ yields $h \circ \phi$ with components $h_{ij}(\phi^k) = h_{ij}(\phi^k(x^a)$ on X and also the pull-back $\phi^* h$ on X with components $(\partial_a \phi^i)(\partial_b \phi^j)(h_{ij}(\phi^k)$. The object $\phi^* h$ is called *first fundamental form* of $\phi : X \to Y$.

We will occasionally write

$$\phi^* h = h(d\phi, d\phi)$$

and also use the bilinear symmetric form $h(\cdot, \cdot)$ with tensorial entries.

Some natural covariant derivative ∇ with components ∇_a is built from g, h,

ϕ according to the following rules.
1- ∇ applied to tensor fields on X equals the Levi-Civita derivative ${}^g\nabla$ to g. For instance,

$$\nabla_a v^c := \partial_a v^c + {}^g\Gamma^c_{ab} v^b,$$

where $v = v^a \partial_a$ is a vector field on X and ${}^g\Gamma^c_{ab}$ are the Christoffel symbols to (g_{ab}).
2- ∇ applied to tensor fields on Y equals some pull-back of the Levi-Civita derivative ${}^h\nabla$ to h. For instance,

$$\nabla_a w^k := (\partial_a \phi^i)(\partial_i w^k + {}^h\Gamma^k_{ij} w^j),$$

where $w = w^i \partial_i$ is a vector field on Y and ${}^h\Gamma^k_{ij}$ are the Christoffel symbols to (h_{ij}). Here $\nabla_a w^k$ is understood to depend on $y^l = \phi^l(x^c)$.
3- There are natural product rules for mixed quantities, the components of which carry both indices $a, b, ...$ and $i, j, ...$ For instance, the *differential* $d\phi$ of $\phi : X \longrightarrow Y$ with components $\partial_a \phi^i$ is a mixed tensor field. The covariant derivative $\nabla d\phi$ of $d\phi$ is called *second fundamental form* of ϕ. It has the components

$$\nabla_a \partial_b \phi^k = \partial_a \partial_b \phi^k - {}^g\Gamma^c_{ab} \partial_c \phi^k + {}^h\Gamma^h_{ij} (\partial_a \phi^i)(\partial_b \phi^j)$$

and the symmetry property

$$\nabla_a \partial_b \phi^k = \nabla_b \partial_a \phi^k.$$

More on calculus for maps between (pseudo-) Riemannian manifolds can be found in the literature, e.g. [4, 5].

3 The field equations

The field theory for g and ϕ considered here is based on the Lagrangian

$$L = \kappa R - e, \tag{3.1}$$

which is the sum of the *gravitational Lagrangian* κR and the *matter Lagrangian* e. Here $R = R[g]$ denotes the scalar curvature of g, $\kappa \neq 0$ is a coupling constant and $e = e[\phi] = e[g, \phi]$ is given by (1.2).
The idea to couple a harmonic map, formerly also called *sigma model*, with Einstein's gravitation appeared in [10, 11, 12] and other early papers.
The following is well-known [10, 11, 12, 13, 14, 15, 16, 17, 18, 7, 8, 9]. We abbreviate $detg := det(g_{ab})$.

Proposition 3.1 *Variation of* $L|detg|^{\frac{1}{2}}$ *with respect to* g *yields the Einstein equation in the form*

$$\kappa(Ric - \frac{1}{2}Rg) = \phi^* h - eg. \tag{3.2}$$

If $m = dimX \geq 3$ *then this is equivalent to*

$$\kappa Ric = \phi^* h. \tag{3.3}$$

Variation of $L|detg|^{\frac{1}{2}}$ *with respect to* ϕ *yields*

$$tr(\nabla d\phi) = 0, \tag{3.4}$$

where $\nabla d\phi$ *is the second fundamental form of* ϕ *and the trace* tr *refers to the metric* g.

The right-hand side of (3.2)

$$T := \phi^* h - eg \tag{3.5}$$

is called *energy-momentum tensor of* ϕ. There holds $e = \frac{1}{2}tr(\phi^* h)$ and (3.5) is equivalent to

$$T - (m-2)^{-1}(trT) = \phi^* h.$$

Proposition 3.2 *From the field equation (3.4) for* ϕ *there follows that* T *is divergence-free:*

$$\nabla^b T_{ab} = 0.$$

The *proof* follows from the identity

$$\nabla^b T_{ab} = h_{ij}(\partial_a \phi^i)(tr\nabla d\phi)^j.$$

4 Energy conditions for a wave map

Let now the metric g have Lorentzian signature $(+ - ... -)$.
Definition 4.1 Let a symmetric two-form $T = T_{ab}dx^a dx^b$ on X be interpreted as an energy-momentum tensor field and let $v = v^a\partial_a$ be a unit timelike vector field on X, i.e., $v_a v^a \equiv g_{ab}v^a v^b = 1$. Then $T(v,v) \equiv T_{ab}v^a v^b$ is called *energy density*, $I := T(\cdot, v)$ with components $I_a := T_{ab}v^b$ is called *momentum one-form*, and $J := I - T(v,v)v$ with components $J_a := I_a - T(v,v)v_a$ is called *proper momentum one-form*.

Physically, v is interpreted as the unit tangent vector to the world line of an observer. This observer measures the quantities $T(v,v)$, I, J.

Every unit timelike vector field $v = v^a \partial_a$ on X gives rise to a positive definite metric $g^+ = g^+_{ab} dx^a dx^b$ on X with components $g^+_{ab} = 2v_a v_b - g_{ab}$. The inverse $(g^{ab}_+) := (g^+_{ab})^{-1}$ has the representation $g^{ab}_+ = 2v^a v^b - g^{ab}$.

Theorem 4.1 *Consider the energy-momentum tensor*

$$T = \phi^* h - eg \tag{4.1}$$

of a map $\phi : X \longrightarrow Y$ between (X,g) and (Y,h). The energy density equals

$$T(v,v) = e_+ := e[\phi, g_+] \equiv \frac{1}{2} g^{ab}_+ (\partial_a \phi^i)(\partial_b \phi^j) h_{ij}. \tag{4.2}$$

It is a positive definite quadratic form in $d\phi$. The momentum one-form I obeys the estimate

$$e^2 \leq |I|^2 \leq e_+^2, \tag{4.3}$$

where

$$|I|^2 := I_a I^a \equiv g^{ab} I_a I_b. \tag{4.4}$$

Proof:
Let us abbreviate $f := \phi^* h$ with components $f_{ab} := (\phi^* h)_{ab} = (\partial_a \phi^i)(\partial_b \phi^j) h_{ij}$. We calculate

$$\begin{aligned} T(v,v) &= T_{av} v^a v^b = (f_{ab} - e g_{ab}) v^a v^b \\ &= f_{ab} v^a v^b - e = \tfrac{1}{2}(2v^a v^b - g^{ab}) f_{ab} = \tfrac{1}{2} g^{ab}_+ f_{ab} = e_+. \end{aligned}$$

The proper momentum one-form J is orthogonal to v, that means $J_a v^a = 0$. Considering that, we find that

$$\begin{aligned} 0 &\leq g^{ab}_+ J_a J_b = -g^{ab} J_a J_b \\ &= -g^{ab} J_a (I_b - e_+ v_b) = -g^{ab} J_a I_b \\ &= -g^{ab}(I_a - e_+ v_a) I_b = -I_a I^a + e_+^2. \end{aligned}$$

Thus, the right-hand side inequality of (4.3) is proved. In order to prove the left-hand side of (4.3), we start with the remark that the matrix (f_{ab}) is positive semi-definite. Let us consider a fixed point $x_0 \in X$ and use coordinates x^a such that

$$v^a = \delta^a_0, \quad g^{ab}_+ = \delta^{ab} \tag{4.5}$$

in that very point x_0. Actually, such coordinates exist. The following 2×2 subdeterminants of (f_{ab}) are non-negative:

$$f_{00}f_{11} - f_{10}f_{10} \geq 0,$$
$$f_{00}f_{22} - f_{20}f_{20} \geq 0,$$
$$\dots$$
$$f_{00}f_{mm} - f_{m0}f_{m0} \geq 0.$$

Let us sum up:

$$f_{00}f_{aa} - f_{a0}f_{a0} \geq 0. \tag{4.6}$$

Here a summation convention applies to the index a and the coordinate conditions (4.5) are assumed. The inequality (4.6) can be brought into a coordinate-invariant form

$$g_+^{ab}(f_{ab}v^c v^d f_{cd} - v^c f_{ac} v^d f_{bd}) \geq 0.$$

Here we insert

$$g_+^{ab} f_{ab} = 2e_+, \quad v^c v^d f_{cd} = e + e_+,$$
$$v^c f_{ac} = v^c (T_{ac} + e g_{ac}) = I_a + e v_a,$$
$$g_+^{ab}(I_a + e v_a)(I_b + e v_b) = 2e_+(e + e_+) + e^2 - |I|^2.$$

Taking all this together, the left-hand side inequality of (4.3) follows.
The conditions

$$T(v, v) \geq 0, \quad |I|^2 \geq 0$$

together form the *dominant energy condition*, which expresses that the energy density is non-negative and that the momentum I is causal. The latter condition physically means that the momentum I propagates with a velocity which is not greater than the velocity of light.
The so-called *strongy energy condition* also holds in the present situation, namely in the form

$$(m - 2)T(v, v) \geq trT.$$

Theorem 4.2 *Consider the energy-momentum tensor $T = T_{ab}dx^a dx^b$ to ϕ as above and lightlike vector fields $l = l^a \partial_a$, $n = n^a \partial_a$ such that $l^a n^a \equiv g_{ab} l^a n^b = 1$. Then*

$$T(l, l) \equiv T_{ab} l^a l^b = h(\phi_* l, \phi_* l) \geq 0, \tag{4.7}$$

and the one-form $I := T(\cdot, l)$ with components $I_a := T_{ab} l^b$ obeys

$$0 \leq |I|^2 \leq 2T(l, l)T(l, n). \tag{4.8}$$

Proof:
Assertion (4.7) follows from

$$T(l,l) = (f_{ab} - eg_{ab})l^a l^b = f_{ab} l^a l^b = h_{ij}(l^a \partial_a \phi^i)(l^b \partial_b \phi^j).$$

The projection tensor p with components

$$p_{ab} := l_a n_b + n_a l_b - g_{ab}$$

is a useful tool. It is orthogonal to l and n, i.e.

$$p_{ab} l^b = p_{ab} n^b = 0,$$

and it is positive semi-definite. Hence

$$0 \le p^{ab} I_a I_b = (2l^a n^b - g^{ab}) I_a I_b = 2T(l,l)T(l,n) - |I|^2,$$

which proves the right-hand side inequality of (4.8)
Below we will also use

$$p^{ab} T_{ab} = (2l^a n^b - g^{ab}) T_{ab} = 2T(l,n) - trT = 2T(l,n) + (m-2)e.$$

Let us, in order to complete the proof, consider a fixed point $x_0 \in X$ and use coordinates x^a such that

$$l^a = l_0^a, \quad n^a = \delta_1^a, \quad (p_{ab}) = diag(0,0,1,...,1)$$

in that very point x_0, where $diag$ indicates a diagonal matrix. Formally the same 2×2 subdeterminants of (f_{ab}) as in the preceding proof are non-negative. Their sum is now in another way transformed into a coordinate-invariant form

$$p^{ab}[f_{ab}(l^c l^d f_{cd}) - (l^c f_{ac})(l^d f_{bd})] \ge 0.$$

Here we insert

$$p^{ab} f_{ab} = p^{ab}(T_{ab} + eg_{ab}) = p^{ab} T_{ab} - (m-2)e = 2T(l,n),$$

$$l^c l^d f_{cd} = T(l,l), \quad p^{ab}(l^c f_{ac})(l^d f_{bd}) = p^{ab} I_a I_b = 2T(l,l)T(l,n) - |I|^2.$$

The result $|I|^2 = g^{ab} I_a I_b \ge 0$ follows.
Physically, theorem (4.2) can be interpreted in terms of a fictional observer which moves faster and faster. In the limit, he reaches the velocity of light and v turns to l. The energy density $T(l,l)$ then remains non-negative and the momentum I remains causal.

Corollary 4.3 *There holds $T(l,n) \geq 0$. Especially, $T(l,n) = 0$ iff $I_a = T(l,l)n_a$.*

Proof:
The formulas (4.7), (4.8) imply $T(l,n) \geq 0$. If $T(l,n) = 0$ then $|I|^2 = 0$ and

$$0 = p^{ab}(l^c f_{bc}) = p^{ab} I_b = (l^a n^b + n^a l^b - g^{ab}) I_b = T(l,l) n^a - I^a$$

5 Implications of the Einstein equation

Let us study

$$\kappa Ric = \phi^* h, \tag{5.1}$$

for given background (Y,h) as a relation between g and ϕ. Contraction with g^{-1} yields

$$\kappa R = 2e. \tag{5.2}$$

Double Contraction with a vector field $v = v^a \partial_a$ on X yields

$$\kappa Ric(v,v) = h(\phi_* v, \phi_* v), \tag{5.3}$$

with the interpretation that $y = \phi(x)$ is to be inserted into the right-hand side of (5.3); $\phi_* v$ denotes the push-forward of v with respect to ϕ. As a conclusion, $\kappa Ric(v,v)$ is positive definite in $\phi_* v$ and is positive semi-definite in v.

Proposition 5.1 *From (5.1) there follows that Ric and $d\phi$, interpreted as linear map, have the same rank:*

$$r := rank(Ric) = rank(d\phi). \tag{5.4}$$

In particular:
$r = 0$ iff ϕ is constant.
$r = m \equiv dimX$ iff ϕ is an immersion.
$r = n \equiv dimY$ iff ϕ is a submersion.
$r = m = n$ iff ϕ is a local diffeomorphism.

The *proof* of (5.4) is an exercise in linear algebra.
Note that $r = 0$ means that (X,g) is Ricci flat, i.e., $Ric = 0$.

Proposition 5.2 *If (5.1) holds then the conditions*

$$Ric(v, v) = 0, \tag{5.5}$$

$$Ric(\cdot, v) = 0, \tag{5.6}$$

$$\phi_* v = 0 \tag{5.7}$$

for a vector field $v = v^a \partial_a$ on X are equivalent to each other. Moreover, they imply

$$\pounds_v Ric = 0, \tag{5.8}$$

where $\pounds_v$ denotes the Lie derivative with respect to v.

Proof:
Equation (5.3) in components reads

$$\kappa R_{ab} v^a v^b = h_{ij} (\phi_* v)^i (\phi_* v)^j,$$

where $(\phi_* v)^i = v^a \partial_a \phi^i$. Moreover, (5.1) implies

$$\kappa R_{ab} v^b = h_{ij} (\partial_a \phi^i)(\phi_* v)^j.$$

These formulas and the definiteness of h give the first assertion. Next, we use comoving coordinates x^a, which are adapted to v, that means $v^a = \delta^a_0$, and we get

$$(\phi^* v)^i = v^a \partial_a \phi^i = \partial_0 \phi^i,$$

$$\kappa \pounds_v R_{ab} = \kappa \partial_0 R_{ab} = \partial_0 (\partial_a \phi^i \partial_b \phi^j h_{ij}(\phi^k)).$$

If $\partial_0 \phi^i = 0$ then also $\partial_0 R_{ab} = 0$. This fact can be translated into the second assertion.

Proposition 5.3 *If the Ricci tensor vanishes on the vectors of some integrable distribution in the tangent bundle TX and (5.1) holds then ϕ is constant on each leaf of the foliation to the distribution.*

Proof:
A distribution of rank s in TX is integrable iff it admits adapted coordinates x^a, which means that the distribution is locally spanned by the coordinate vector fields $\partial_1, \partial_2, ..., \partial_s$. The assumption becomes

$$Ric(\partial_a, \partial_b) = 0 \quad for\ a, b = 1, 2, ..., s.$$

Proposition (5.2) then implies

$$\partial_1\phi^i = 0,\ \partial_2\phi^i = 0,\ ...,\ \partial_s\phi^i = 0.$$

Hence ϕ^i does not depend on $x^1, x^2, ..., x^s$ and is constant if the point x varies in a leaf of the foliation, i.e., if $x^{s+1} = const.,\ ...,\ x^m = const.$

Proposition 5.4 *The Einstein equation (5.1) implies*

$$\kappa(\nabla_a R_{bc} + \nabla_b R_{ca} - \nabla_c R_{ab}) = 2h_{ij}(\nabla_a\partial_b\phi^i)(\partial_c\phi^j). \tag{5.9}$$

Proof:
Covariant differentiation of (5.1) gives

$$\kappa\nabla_c R_{ab} = h_{ij}[(\nabla_c\partial_a\phi^i)(\partial_b\phi^j) + (\partial_a\phi^i)(\nabla_c\partial_b\phi^j)].$$

Some rearrangement yields (5.9).

Proposition 5.5 *The Einstein equation (5.1) implies*

$$h(tr(\nabla d\phi), d\phi) = 0. \tag{5.10}$$

This equation for a submersion ϕ implies the harmonic or wave map equation (3.4).

Proof:
The Einstein tensor $Ric - \frac{1}{2}Rg$ is divergence-free. This fact and (5.9) imply

$$2h(tr(\nabla d\phi), d\phi)_c \equiv 2h_{ij}tr(\nabla d\phi)^i(\partial_c\phi^j) = \kappa g^{ab}(\nabla_a R_{bc} + \nabla_b R_{ca} - \nabla_c R_{ab}) = 0.$$

If ϕ is a submersion, then the matrix with elements $h(\cdot, d\phi)_{ic} = h_{ij}\partial_c\phi^j$ has maximal rank and therefore (5.10) implies (3.4).

Proposition 5.6 *If the Ricci tensor is covariantly constant, i.e., $\nabla Ric = 0$, and the Einstein equation (5.1) holds for a submersion ϕ, then ϕ is totally geodesic, that means*

$$\nabla d\phi = 0. \tag{5.11}$$

Proof:
If $\nabla Ric = 0$ then (5.9) reduces to $h_{ij}(\nabla_a\partial_b\phi^i)(\partial_c\phi^j) = 0$. If, additionally, ϕ is a submersion, then (5.11) follows by means of the rank argument already

used in the preceding proof.

The next proposition needs some preparation. A diffeomorphism $f : X \longrightarrow X$ is called a *Ricci symmetry* iff

$$f^* Ric = Ric. \tag{5.12}$$

A vector field $v = v^a \partial_a$ on X is called an *infinitesimal Ricci symmetry* or a *Ricci collineation* iff

$$\pounds_v Ric = 0. \tag{5.13}$$

It is well known that the flow f_t for $t \in I$ of a Ricci collineation v is a one-parameter family of Ricci symmetries, i.e. $f_t^* Ric = Ric$. Here $x = f_t(x_0)$ by definition represents the solution of the initial-value problem

$$\frac{dx}{dt} = v(x), \quad x|_{t=0} = x_0$$

and I denotes an open interval which contains 0.

Proposition 5.7 *Let $v = v^a \partial_a$ be a Ricci collineation of (X, g). Then the Einstein equation (5.1) implies that $\phi : X \longrightarrow Y$ is invariant under the flow f_t for $t \in I$ of v, that means*

$$\phi \circ f_t = \phi. \tag{5.14}$$

Proof:
Let us again use comoving coordinates such that $v^a = \delta^a_0$. In these coordinates, f_t is expressed by a translation $x^0 \mapsto x^0 + t$, $x^1 \mapsto x^1$, ..., $x^{m-1} \mapsto x^{m-1}$. We know already from the proof of proposition (5.2) $\partial_0 \phi^i = 0$, i.e. each ϕ^i is independent of x^0. Hence $\phi^i = \phi^i(x^a)$ does not change under $x^0 \mapsto x^0 + t$, which is just expressed by (5.14).

The following definition is useful.
Definition 5.1 A property of subsets of a manifold X holds *globally* if it is valid for X. It holds *locally* if every point $x_0 \in X$ has a neighborhood U such that the property is valid for U.

Theorem 5.8 *Let $\phi : X \longrightarrow Y$ be a submersion between smooth manifolds X, Y and let Y be equipped with a positive definite metric h.*
1- Locally there is a positive definite metric g on X such that ϕ becomes a harmonic map (X, g) and (Y, h).
2- Locally there is a Lorentzian metric g on X such that ϕ becomes a wave map.

Proof:
We consider (5.1) and use the fact that the *problem of prescribed Ricci curvature* is locally solvable in the two cases [19, 20]. More precisely: cf., eg.
1- Einstein's equation (5.1) locally has a positive definite solution g. It can be constructed, e.g., through some boundary value problem [19]. By assumption, ϕ is a submersion; proposition 5.5 implies $tr(\nabla d\phi) = 0$.
2- Einstein's equation (5.1) locally has a Lorentzian solution g. It can be constructed, e.g., through some Cauchy initial value problem cf., e.g. [20]. An argument like in item 1 completes the proof.

6 Product and warped product source manifolds

Definition 6.1: The *product* (X, g) of two (pseudo-) Riemannian manifolds $('X, 'g)$, $(''X, ''g)$ is given by $X = 'X \times ''X$ as a product of manifolds and by

$$g(u, v) = 'g('u, 'v) + ''g(''u, ''v)$$

where u, v are vector fields on X, $'u, 'v$ are the push-forwards of u, v with respect to the projection $X \longrightarrow 'X$, and $''u$, $''v$ are the push-forwards of u, v with respect to the projection $X \longrightarrow ''X$.
We write then

$$g = 'g \oplus ''g,$$

$$dim'X = 'm, \quad dim''X = ''m, \quad m = 'm + ''m.$$

We apply the index convention

$$'a, 'b, 'c, ... = 1, 2, ..., 'm; \quad ''a, ''b, ''c, ... = 'm + 1, 'm + 2, ..., m.$$

Definition 6.2: The *warped product* (X, g) is given by $X = 'X \times ''X$ like above and by

$$g(u, v) = 'g('u, 'v) + w^2\, ''g(''u, ''v),$$

where the *warping function* w is a map $'X \longrightarrow \Re$ with positive values $w > 0$.
We write then

$$g = 'g \oplus w^2\, ''g,$$

The following is known.

Proposition 6.1 *If (X, g) is the product of $('X, 'g)$, $(''X, ''g)$ then the Einstein equation $\kappa Ric = \phi^* h \equiv h(d\phi, d\phi)$ decomposes into the two Einstein equations*

$$\kappa' Ric = h('d\phi, 'd\phi), \quad \kappa'' Ric = h(''d\phi, ''d\phi) \tag{6.1}$$

and the orthogonality condition with respect to h:

$$h('d\phi, ''d\phi) = 0, \tag{6.2}$$

in components

$$\kappa\, 'R_{'a\,'b} = h_{ij}(\partial_{'a}\phi^i)(\partial_{'b}\phi^j), \quad \kappa\, ''R_{''a\,''b} = h_{ij}(\partial_{''a}\phi^i)(\partial_{''b}\phi^j), \tag{6.3}$$

$$h_{ij}(\partial_{'a}\phi^i)(\partial_{''b}\phi^j) = 0 \tag{6.4}$$

Proposition 6.2 *If (X, g) is the warped product of $('X, 'g)$, $(''X, ''g)$ with warping function w then the Einstein equation $\kappa Ric = \phi^* h \equiv h(d\phi, d\phi)$ decomposes into*

$$\kappa('Ric - ''mw^{-1}\,'\nabla' dw) = h('d\phi, 'd\phi), \tag{6.5}$$

$$\kappa(''Ric - \frac{1}{''m} w^{2-''m}\; ''\Delta w^{''m}\, ''g) = h('d\phi, ''d\phi), \tag{6.6}$$

$$h('d\phi, ''d\phi) = 0. \tag{6.7}$$

The trace parts of (6.5), (6.6) read

$$\kappa('R - ''mw^{-1}\,'\Delta w) = 2\, 'e, \tag{6.8}$$

$$\kappa(''R - w^{2-''m}\; '\Delta w^{''m}) = 2\, ''e. \tag{6.9}$$

The proof is an exercise in higher differential geometry

We say that $\phi : 'X \times ''X \longrightarrow Y$ does not depend on $'x$ iff the restriction $\phi(\cdot, ''x) : 'X \longrightarrow Y$ is a constant map for every $''x \in ''X$. We say that ϕ does not depend on $''x \in ''X$ in the analogous situation.

Theorem 6.3 *Let in the situation of proposition 6.2 the restriction $\phi('x, \cdot) : ''X \longrightarrow X$ be a submersion for every $'x \in 'X$. Then ϕ does not depend on $'x$.*

Proof:

If every $\phi('x, \cdot)$ is a submersion then the quantities $h_{ij}\partial_{''b}\phi^j$ in (6.4) form a matrix of maximal rank, and (6.4) implies $\partial_{'a}\phi^i = 0$, which gives the assertion.

Theorem 6.4 *Let in the situation of proposition 6.2 the first factor* $('X, 'g)$ *be properly Riemannian and closed. Then the following holds.*
(i) The symmetric two-form $\kappa''Ric$ *is positive semi-definite.*
(ii) If $\kappa''Ric$ *is not everywhere positive then* $w = const.$
(iii) $\kappa \int_{'X} w\, 'R d'vol \geq 0.$
(iv) If $\int_{'X} w\, 'R d'vol = 0$ *then* $w = const.$ *and* ϕ *does not depend on* $'x$.

Proof:
(i) Multiply (6.6) by $w^{''m-2}$ and evaluate the two-forms at a vector field $''v \neq 0$ on $''X$:

$$\kappa[w^{''m-2}\, ''Ric(''v, ''v) - \frac{1}{''m}('\Delta w^{''m})|''v|^2] = w^{''m-2}h(''d\phi(''v), ''d\phi(''v)). \tag{6.10}$$

Integrate this equation over $'X$; the Laplacian term does not contribute, hence

$$\kappa''Ric(''v, ''v)\int_{'X} w^{''m-2}d'vol = \int_{'X} w^{''m-2}h(''d\phi(''v), ''d\phi(''v))d'vol.$$

Both the integrals are non-negative, hence $\kappa''Ric(''v, ''v) \geq 0$ for every $''v$.
(ii) If $''Ric(''v, ''v)(''x_0) = 0$ for some point $''x_0 \in ''X$ then

$$\int_{'X} w^{''m-2}h(''d\phi(''v(''x_0)),\ ''d\phi(''v('x_0))d'vol = 0,$$

which implies $''d\phi(''v(''x_0)) = 0$. Evaluation of (6.10) at the point $''x_0$ reduces (6.10) to

$$('\Delta w^{''m})|''v(''x_0)|^2 = 0.$$

We can assume $|''v(''x_0)|^2 \neq 0$; hence $w^{''m}$ is a harmonic function. A harmonic function on a closed manifold is constant.
(iii) Multiply (6.8) by w and integrate then over $'X$. The Laplacian term does not contribute; hence

$$\kappa \int_{'X} w\, 'R d'vol = \int_{'X} v\, 'e d'vol \geq 0.$$

(iv) The last integral vanishes only if $'e = 0$, which implies $'d\phi = 0$, hence $\phi(\cdot, ''x) = const$ for fixed $''x \in ''X$. But then $''e(\cdot, ''x) = const$ and the standard separation argument can be applied to

$$\kappa w^{2-'m}\, '\Delta w^{''m} = \kappa\, ''R - 2\, ''e. \tag{6.11}$$

Thus both sides of (6.11) equal a constant c. Integration of

$$\kappa\,'\Delta w''^{m} = c w''^{m-2}$$

yields $c = 0$. Hence w''^{m} is a harmonic function on the closed manifold $'X$. We find $w''^{m} = const$, $w = const$.

Proposition 6.5 *Let in the situation of the preceding theorem the second factor $(''X, ''g)$ be properly Riemannian with vanishing scalar curvature: $''R = 0$. Then $w = const$ and ϕ does not depend on $''x \in {''X}$.*

Proof:
Now equation (6.9) reduces to

$$-\kappa'\Delta w''^{m} = 2w''^{m-2}\,''e$$

and we have $''e \geq 0$. Integration over the closed manifold $'X$ yields

$$0 = \int_{'X} w''^{m-2}\,''e d'vol,$$

which implies

$$''e \equiv {''tr}\, h(''d\phi, ''d\phi) = 0.$$

Hence $''d\phi = 0$, i.e. ϕ does not depend on $''x$, and w''^{m} becomes a harmonic function. We arrive at $w''^{m} = const$, $w = const$.

Example 6.1. Let in the situation of proposition 6.2 the first factor $('X, 'g)$ be the unit circle S^1. It is a flat properly Riemannian closed manifold. Theorem 6.4 implies that $w = const$ and ϕ does not depend on $'x$.

Example 6.2. A static metric $g = w^2 dt^2 - {'g}$ can be interpreted as a warped product metric on $X = {'X} \times {''X}$ where there the second factor $''X$ is one-dimensional. The proof of proposition 6.5 works, with a slight modification, for this case. Hence Einstein's equation $\kappa Ric = \phi^* h$ implies that $w = const$ and ϕ does not depend on $t \in {''X}$.

7 Source manifolds with radiation conditions

Lichnerowicz [21] proposed *conditions of pure radiation* for a Lorentzian manifold (X, g):

$$R_{abcd}l^d = 0, \quad R_{ab[cd}l_{e]} = 0,$$

where R_{abcd} are the components of the Riemann curvature tensor and $l = l^a\partial_a$ is a lightlike vector field. Bel [22] proposed weaker conditions:

$$R_{abcd}l^bl^d = 0, \quad l^bR_{ab[cd}l_{e]} = 0, \quad l_{[f}R_{ab][cd}l_{e]} = 0.$$

There are also radiation conditions for the Ricci tensor Ric or for the *Einstein tensor* $G := Ric - \frac{1}{2}Rg$ with components

$$G_{ab} := R_{ab} - \frac{1}{2}Rg_{ab},$$

namely

$$R_{ab}l^b = 0, \quad R_{a[b}l_{c]} = 0,$$

$$G_{ab}l^b = 0, \quad G_{a[b}l_{c]} = 0.$$

One of the present authors studied radiation conditions in [23, 24].

Theorem 7.1 *If the Einstein equation* $\kappa Ric = \phi^*h$ *holds then the conditions*

$$G_{ab}l^b = 0, \tag{7.1}$$

$$R_{a[b}l_{c]} = 0 \tag{7.2}$$

for a lightlike vector field $l = l^a\partial_a$ *are equivalent to each other.*

Proof:
There is another lightlike vector field $n = n^a\partial_a$ such that $l^an_a = 1$. Then the sum $v := l + n$ is timelike with $v^av_a = 2$ and $g^+_{ab} = v_av_b - g_{ab}$ are the components of a positive definite metric g_+.
Let us consider

$$\psi^i_a := 2v^bl_{[a}\partial_{b]}\phi^i \equiv l_a(\phi_*v)^i - \partial_a\phi^i$$

and calculate

$$g^{ab}_+h_{ij}\psi^i_a\psi^j_b = -g^{ab}h_{ij}\psi^i_a\psi^j_b = h_{ij}(\phi_*v)^i(\phi_*l)^j - e = \kappa G_{ab}v^al^b.$$

Here we made use of

$$g^{ab}_+ = v^av^b - g^{ab}, \quad v^a\psi^i_a = 0$$

and of $\kappa Ric = \phi_*h$. If $G_{ab}l^b = 0$ then the above positive definite expression vanishes and we get

$$\psi^i_a = 0, \quad \partial_a\phi^i = l_a(\phi_*v)^i,$$

$$\kappa R_{ab} = h_{ij}(\partial_a \phi^i)(\partial_b \phi^j) = l_a l_b h(\phi_* v, \phi_* v).$$

Conversely, from $R_{ab} = r l_a l_b$ with some scalar r there follow $R = 0$ and (7.1).

We here dwell again on the fact that to every smooth vector field $v = v^a \partial_a$ on X there are comoving coordinates $x^a = x^0, x^1, ..., x^{m-1}$, which means $v^a = \delta^a_0$. The adaptedness of coordinates is preserved by coordinate transformations of the form

$$\bar{x}^0 = x^0 + f(x^i), \quad \bar{x}^i = \bar{x}^i(x^j). \tag{7.3}$$

We apply in this section the index convention

$$a, b, c, ... = 0, 1, 2, ..., m-1;$$

$$i, j, k, ... = 1, 2, ..., m-1;$$

$$I, J, K, ... = 2, 3, ..., m-1.$$

Proposition 7.2 *A Lorentzian manifold (X, g) admits a lightlike*
... hypersurface-orthogonal Killing vector field $l = l^a \partial_a$, that means

$$l_{[c} \nabla_a l_{b]} = 0, \quad \nabla_{(a} l_{b)} = 0,$$

iff in coordinates adapted to l the metric assumes the form

$$g = 2 g_{01} dx^0 dx^1 + g_{ij} dx^i dx^j, \tag{7.4}$$

where g_{01}, g_{ij} do not depend on x^0. The component $g_{01} = g_{01}(x^k)$ is invariant under gauge transformations (7.3) and the part $g_{IJ} dx^I dx^J$ of (7.4) shows tensorial behavior under the part $\bar{x}^i = \bar{x}^i(x^j)$ of (7.3). Moreover, the matrix $(g_{IJ}) = (g_{IJ}(x^k))$ is negative definite.
... covariantly constant vector field $l = l^a \partial_a$, that means $\nabla_a l_b = 0$, iff in coordinates adapted to l the metric assumes the form

$$g = 2 dx^0 dx^1 + g_{ij} dx^i dx^j, \tag{7.5}$$

where the components g_{ij} do not depend on x^0.
... covariantly constant vector field $l = l^a \partial_a$ such that the Bel condition

$$l_{[e} R_{ab][cd} l_{f]} = 0 \tag{7.6}$$

holds iff there are coordinates adapted to l such that

$$g = 2 g_{01} dx^0 dx^1 + g_{11} (dx^1)^2 + 2 g_{1I} dx^1 dx^I - dx^I dx^I, \tag{7.7}$$

where g_{11}, g_{1I} do not depend on x^0 and summation over I is applied.
... covariantly constant vector field $l = l^a \partial_a$ such that the Lichnerowicz condition

$$l_{[e} R_{ab]cd} = 0 \tag{7.8}$$

holds iff there are coordinates adapted to l such that

$$g = 2g_{01} dx^0 dx^1 + g_{11}(dx^1)^2 - dx^I dx^I, \tag{7.9}$$

where g_{11} does not depend on x^0 and summation over I is applied.

All these facts together with *proof* and additional information are given in the papers [24, 23].

A Lorentzian manifold (X, g) which admits a covariantly constant light-like vector $l = l^a \partial_a$ is called a *plane-fronted gravitational wave with parallel rays*, abbreviated *pp-wave*. Note that from $\nabla_a l_b = 0$ and the Ricci identity there follows the Lichnerowicz condition

$$R_{abcd} l^d = 0.$$

Theorem 7.3 *A metric (7.4) satisfies an Einstein equation $\kappa Ric = \phi^* h$ iff $g_{01} = g_{01}(x^1, x^K)$ is a harmonic function of $x^K = x^2, x^3, ..., x^{m-1}$ with respect to the positive definite metric (which depends on x^1 as a parameter) $-g_{IJ} dx^I dx^J$.*

Proof:
Some calculation gives the components

$$R_{00} = 0, \quad R_{01} = \frac{1}{2} \Delta g_{01}$$

of the Ricci tensor $Ric = R_{ab} dx^a dx^b$, where Δ denotes the Laplace operator with respect to $-g_{IJ} dx^I dx^J$. By proposition 5.2, from $R_{00} \equiv R_{ab} l^a l^b = 0$ and $\kappa Ric = \phi^* h$ there follows $R_{01} \equiv R_{1b} l^b = 0$. The assertion follows.

Theorem 7.4 *Let $m = dim X = 4$. A metric of the form (7.7) satisfies an Einstein equation $\kappa Ric = \phi^* h$ iff it satisfies (7.8), that means iff it can be brought into the form (7.9).*

Proof:
Calculation of Ricci components gives $R_{IJ} = 0$; in particular $R_{II} = 0$ (without summation). By Proposition 5.2, from this and $\kappa Ric = \phi^* h$ there follows

$R_{1I} = 0$. For $m = 4$, there are only two independent curvature components of type R_{iJKL}, namely

$$R_{1223} = -R_{13}, \quad R_{1323} = R_{12}.$$

Thus we get $R_{iJK2} = 0$ which is expressed by (7.9) in a coordinate invariant way.

8 Discussion

The literature on harmonic or wave maps is very extensive. There are good surveys on harmonic maps [4, 5]. Work on such maps in the role of matter fields coupled with gravitation began about 1980 [10, 11]. One of the authors of this paper worked, with coauthors, already on this subject; we refer to the paper [17] and the unpublished preprint [25].

The Einstein equation $\kappa Ric = \phi^* h$ or $\kappa G = T$, where $G = Ric - \frac{1}{2}Rg$ denotes the Einstein tensor and $T = \phi^* h - eg$ the energy-momentum tensor of ϕ, exhibits some remarkable properties:
- The rank of Ric, taken as a linear map, equals the rank of the differential $d\phi$ (Proposition 5.1).
- The symmetries of the Ricci tensor Ric and of the map ϕ are closely related to each other (Propositions 5.2, 5.3, 5.7).
- If ϕ is submersive then the Einstein equation implies the harmonic or wave map equation (Proposition 5.5; cf. also Proposition 5.6).
- In the Lorentzian case there are identities and estimates for the energy-momentum tensor T which indicate a physically good behavior of T (Proposition 3.2, Theorems 4.1, 4.2).
- In the Lorentzian case there is a tendency to enhance radiation conditions. That means, the Einstein equation leads from one condition to a stronger condition (Section 7).

There is also a situation where the Einstein equation serves as an auxiliary construction: for a given submersion ϕ there locally exists a metric g on X which makes ϕ to a harmonic or wave map (Theorem 5.8).
One paper cannot touch all aspects of a subject. We did not discuss here:
- Bochner-Weitzenböck technique [5, 13, 14, 15],
- consequences of second variation formulas,
- factorizations of the map ϕ [18],
- exact solutions [10, 7, 8, 9, 12, 13, 14, 15, 16, 11],
- coupling of ϕ to a gravitational theory other than Einstein's theory.

These topics are by far not exhausted and could be subjects of further research.

Acknowledgments

Former discussions and cooperations with T. Hirschmann and T. Deck are gratefully acknowledged. We thank the professors G. Huisken and A. D. Rendall for inspiring discussions.

References

[1] A. Macias, J. L. Cervantes-Cota and C. Lämmerzahl, Exact Solutions and Scalar Fields in Gravity, Kluwer, Dordrecht (2001).

[2] V. Faraoni, Cosmology in Scalar-Tensor Gravity, Kluwer, Dordrecht (2004).

[3] A. D. Rendall, Partial Differential Equations in General Relativity, Oxford Univ. Press (2008).

[4] J. Eells and J. H. Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. Math. **86** (1964), 109-160.

[5] J. Eells and L. Lemaire, Two Reports on Harmonic Maps, Worls Scientific, Singapore (1993).

[6] M. Struwe, Wave maps, Birhhauser, Basel (1997).

[7] P. Bizon, T. Chmaj and Z. Tabor, Dispersion and collapse of wave maps. Nonlinearity **13** (2000), 1411-1423.

[8] P. Bizon and A. Wasserman, Self-similar spherically symmertic wave maps coupled to gravity, Phys. Rev. D **62** (2000), 084031.

[9] P. Bizon and A. Wassermann, On the existence of self-similar spherically symmetric wave maps coupled to gravity, Class. Quant. Grav. **19** (2002), 3309-3321.

[10] V. de Alfaro, S. Fubini and G. Furlan, Gauge theories and strong gravity, Il Nuovo Cim. A **50**, (1979), 523-554.

[11] C. Omero and R. Percacci, Generalized non-linear σ-models in curved space and spontaneous compactification, Nucl. Phys. B **165** (1980), 351-364.

[12] M. Gell-Mann and B. Zwiebach, Spacetime compactification induced by scalars, Phy. Lett. B **141** (1984), 333-336.

[13] G. Ghika, Harmonic maps and submersions in local Euclidean gravity coupled to the σ-model, Rev. Roum. Phys. **31** (1986), 635-648.

[14] G. Ghika and A. Corciovei, Static solutions for the sigma-model coupled to gravity, Rev. Roum. Phys. **32** (1987), 827-835.

[15] G. Ghika and M. Visinescu, Four-dimensional σ-model coupled to the metric tensor field, Il Nuovo Cim. B **59** (1980), 59-73.

[16] S. Ianus and M. Visinescu, Spontaneous compactification induced by non-linear dynamics, Class. Quant. Grav. **3** (1986), 889-896.

[17] R. Schimming and T. Hirschmann, Harmonic maps from spacetimes and their coupling to gravitation, Astron. Nachr. **309** (1988), 311-321.

[18] A. P. Whitman, R. J. Knill and W. R. Stoeger, Some harmonic maps on pseudo-Riemannian manifolds, Internat. J. Theoret. Phys. **25** (1986), 1139-1153.

[19] D. DeTurck, Existance of metrics with prescribed Ricci curvature: local theory, Invent. Math. **65** (1981), 179-207.

[20] H. Friedrich and A. D. Rendall, The Cauchy Problem for the Einstein Equations, Springer, Berlin (2000).

[21] A. Lichnerowicz, Radiations en relativite generale, In Colloque de Royaumont (1959). Edition CNRS, Paris (1962).

[22] L. Bel, La radiation gravitationalle. In: Colloque de Royaumont (1959). Edition CNRS, Paris (1962).

[23] R. Schimming, Riemannsche Räume mit ebenfrontiger und mit ebener Symmetrie, Math. Nachr. **59** (1974), 129-162.

[24] B. Fiedler and R. Schimming, Exact solutions of the Bach field equations of general relativity, Rep. Math. Phys., **17** (1980), 15-36.

[25] T. Deck and R. Schimming, Harmonic maps coupled to the Einstein equation, Preprint Universität Mannheim (1991).