Foundations of Physics Letters, Vol. 8, No. 6, 1995

SCHWARZSCHILD GEOMETRY, ONCE MORE

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Received December 27, 1994; revised July 19, 1995

Schwarzschild geometry exhibits interesting features when the field equations are decomposed with respect to a system of freely falling observers. We only use quantities behaving like tensors under a restricted group of transformations of the reference system. Moreover Lorentz transformations with non-constant velocities drastically change the physical picture of the theory.

Key words: Schwarzschild geometry, tidal forces, gravitation, field equations, generalized Lorentz transformation, conservation laws.

1. INTRODUCTION

The Schwarzschild solution describing a static spherical symmetric gravitation field was the first solution of the Einstein equation. Although it has been explored for more than seven decades it is still topical.

In our opinion, gravitation theory suffers from being presented in a highly coordinate dependent manner and has only marginal similarity to other field theories. We try to improve the situation by using tetrad formalism and quantites behaving as tensors under restricted transformations of locally defined tetrad fields. These tensors are the field strength, energy and stresses of the gravitation field. As the field equations are non-linear it can be shown that quadratic constructions of these tensors are responsible for effects of selfgravitation. Under more general transformations the quanties mentioned above are subjected to an inhomogeneous transformation law and the physical picture of the theory changes considerably.

2. THE FREELY FALLING SYSTEM

In their textbook, Misner, Thorne, and Wheeler [1] illustrate in detail all the tortuous influences a freely falling observer is subjected to. In the radial direction the observer will be stretched because of the gradient of the gravitation field (tidal forces [2]). Also he will be squeezed, because the trajectories of the field are converging to the center of attraction. We will now describe these forces in detail. We strictly reject the use of global coordinates and use local tetrad fields instead. This simplifies many calculations and makes all quantities accessible to an immediate physical interpretation. A 4-bein consists of rods and clocks and the components of a tensor are values measured by these rods and clocks [3]. The tetrads $e^i_{\ a}(a = 1, \ldots, 4$ is the index labelling vectors and $i = 1, \ldots, 4$ denote the holonomic coordinates of these vectors) are comoving with the freely falling observers. The fourth of these vectors is timelike, and its coordinate invariant components are

$$t_a = e^4_i e^i_a = \{0, 0, 0, 1\}.$$

A second tetrad is static. It is tied to three-dimensional space enclosing the center of attraction. Its fourth member

$$u_m = e^4_i e^i_m = \{0, 0, 0, 1\},$$

$m = 1^\bullet, \ldots, 4^\bullet$, is timelike and orthogonal to the other vectors [4]. Both systems are related by a generalised Lorentz transformation (a transformation with nonconstant velocity parameters) operating on the tetrad components. These two kinds of systems are relatively accelerated, and the components of the transformation matrix are

$$A^1_{4^\bullet} = A^{4^\bullet}_1 = i\alpha v, A^4_{1^\bullet} = A^{1^\bullet}_4 = -i\alpha v, A^1_{1^\bullet} = A^{1^\bullet}_1 = A^4_{4^\bullet} = A^{4^\bullet}_4 = \alpha,$$
$$v = v(r) = -\sqrt{2M/r}, \ \alpha = 1/\sqrt{1 - 2M/r}. \tag{2.1}$$

Using Schwarzschild's spherical coordinates $(r, \vartheta, \varphi, t)$ for both systems, the components of the static 4-bein are $(\sigma = r \sin \vartheta)$:

$$\begin{aligned} e^1_{1^\bullet} &= 1/\alpha, \ e^2_{2^\bullet} = 1/r, \ e^3_{3^\bullet} = 1/\sigma, \ e^4_{4^\bullet} = \alpha, \\ e^{1^\bullet}_1 &= \alpha, \ e^{2^\bullet}_2 = r, \ e^{3^\bullet}_3 = \sigma, \ e^{4^\bullet}_4 = 1/\alpha. \end{aligned} \tag{2.2}$$

With the help of (2.1, 2.2), we are able to calculate the timelike derivatives with respect to the freely falling system,

$$\Phi_{|4} = A^{1^\bullet}_4 \Phi_{|1^\bullet} = -iv\frac{\partial}{\partial r}\Phi, \quad \Phi_{|4} = -i\Phi^\bullet, \quad \Phi^\bullet = (v\,\text{grad})\Phi.$$

The relative velocity of the two systems is $v(r) = r^*$ and is pointing towards the center of acceleration. Defining a covariant derivative with respect to the freely falling system,

$$\Phi_{a\|b} = \Phi_{a|b} - A_{ba}{}^{c}\Phi_{c},$$

and a second one with restricted transformation properties (rotations in the three-dimensional subspace)

$$\Phi_{a\wedge b} = \Phi_{a|b} - {}^{*}A_{ba}{}^{c}\Phi_{c}, A_{ab}{}^{c} = {}^{*}A_{ab}{}^{c} + D_{ab}{}^{c}, \Phi_{a\|b} = \Phi_{a\wedge b} - D_{ba}{}^{c}\Phi_{c}, \tag{2.3}$$

one is able to calculate the connection coefficients using well-known methods of differential geometry:

$$\begin{gathered} {}^{*}A_{ab}{}^{c} = b_a B_b b^c - b_a b_b B^c + \ell_a S_b \ell^c - \ell_a \ell_b S^c, D_{ab}{}^{c} = D_{a}{}^{c} t_b - D_{ab} t^c, \\ b_a = \{0,1,0,0,\}, \ell_a = \{0,0,1,0\}. \end{gathered} \tag{2.4}$$

The quantities

$$B_a = \{1/r, 0,0,0\}, S_a = \{1/r, (1/r)\cot\vartheta, 0, 0\} \tag{2.5}$$

are the conventional spherical constituents of the three-dimensional part of the flat connection,

$$D_{ab} = t_{a\|b} = -D_{ba}{}^{c} t_c, D_{[ab]} = 0, \quad t^a D_{ab} = 0, D_{ab} t^b = 0 \tag{2.6}$$

are the deformation field strength [5] formerly mentioned. They are a consequence of the curvature of time. The only components are

$$D_{11} = \frac{i\dot{v}}{2r}, \quad D_{22} = -\frac{i\dot{v}}{r}, D_{33} = -\frac{i\dot{v}}{r}. \tag{2.7}$$

The curvature tensor of this connection is

$$\begin{aligned} R_{dab}{}^{c} &= 2\left[A_{[a\bullet b\bullet}{}^{c}{}_{\|d]} - A_{[a\bullet b\bullet}{}^{g} A_{d]g}{}^{c}\right] = 2\left[D_{[a\bullet b\bullet}{}^{c}{}_{\wedge d]} - D_{[da]}{}^{g} D_{gb}{}^{c} - D_{[d\bullet b\bullet}{}^{g} D_{a]g}{}^{c}\right] \\ &= 2\left[D_{[a}{}^{c}{}_{\wedge d]} t_b - D_{[a\bullet b\bullet \wedge d]} t^c + t_{[a} D_{d]}{}^{g} D_{g}{}^{c} t_b + t_{[a} D_{d]}{}^{g} D_{gb} t^c + D_{b[d} D_{a]}{}^{c}\right], \end{aligned} \tag{2.8}$$

and the identity $R_{[dab]}{}^{c} = 0$ leads to

$$D_{[a}{}^{c}{}_{\wedge d]} t_b + D_{[b}{}^{c}{}_{\wedge a]} t_d + D_{[d}{}^{c}{}_{\wedge b]} t_a = 0 \tag{2.9}$$

or, multiplicated with t^b, to

$$D_{[a\wedge\underline{b}]}^{\ c} = 0, \tag{2.10}$$

the bar denoting projection on the three-dimensional space (the set of hyperplanes orthogonal to t^b). In (2.8) we have made use of the fact, that the three-dimensional space is flat:

$$^*R_{\alpha\beta\gamma}{}^{\delta} = -2\left[{}^*A_{[\alpha\bullet\gamma\bullet\wedge\beta]}^{\ \ \delta} - {}^*A_{[\alpha\bullet\gamma\bullet}^{\ \ \varepsilon}\ {}^*A_{\beta]\varepsilon}{}^{\delta}\right] = 0, \quad \alpha = 1,2,3.$$

The four-dimensional extension of this quantity including the time derivative is not flat, of course:

$$^*R_{dab}^{\ \ c} = 2D_{[da]}^{\ \ g}\ {}^*A_{gb}{}^{c};$$

and by contraction we get the Ricci tensor

$$^*R_{ab} = 2D_{[ca]}^{\ \ g}\ {}^*A_{gb}{}^{c}. \tag{2.11a}$$

We now contract (2.8) and simplify this new relation with (2.11a), and then we split into space-like and time-like parts:

$$\begin{aligned} R_{\underline{ab}} &= -D_{ab\wedge c}t^c - D_{ab}D_c^{\ c}, \\ R_{\underline{a}b}t^b &= \ \ D_{a\wedge c}^{\ c} - D_{c\wedge \underline{a}}^{\ c}, \\ R_{ab}t^a t^b &= -D_{c\wedge a}^{\ c}t^a - D_{ab}D^{ab}. \end{aligned} \tag{2.11b}$$

Introducing the quantity

$$\Theta_{ab} = D_{ab\wedge c}t^c + D_a^{\ c}D_{bc}, \tag{2.12}$$

we get with help of the vacuum field equations

$$\Theta_{ab} = \kappa t_{ab}, \kappa t_{ab} = D_a^{\ c}D_{cb} - D_c^{\ c}D_{ab}, \tag{2.13}$$

which has a well defined meaning. The deviation δx^a of the time-like geodesics $t^b t_{a\|b} = 0$ may be calculated by

$$e_i^{\ c}\frac{D^2\delta x^i}{Ds^2} = e_i^{\ c}(\delta x^i)_{\|ab}t^a t^b = -R_{dab}^{\ \ \ c}\delta x^d t^a t^b = \Theta_a^{\ c}\delta x^a.$$

With $\delta x^c = \{\delta r, r\delta\vartheta, r\sin\vartheta\delta\varphi\}$, the components

$$\Theta_{ab} = \{-2M/r^3, M/r^3, M/r^3, 0\}$$

are regular, except at $r = 0$, and describe the relative accelerations of freely falling points of unit mass. They cause a stretching in the radial direction and a squeezing in the two other spacelike directions. (See also the example in the textbook mentioned above [1].) The accelerations are in equilibrium with the stresses [1]

$$\kappa t_{ab} = 2D_{[a}{}^{c}D_{c]b}. \tag{2.14}$$

This quantity is a symmetric traceless tensor with respect to a restricted transformation and is locally conserved:

$$t_{a\wedge b}{}^{b} = 0, \tag{2.15}$$

which may be proved with help of (2.11b) and $D^{ab}D_{ab} = (D_c^{\ c})^2$. The relations (2.14) and (2.15) are of course destroyed by more general transformations. What happens in performing (2.1) will be discussed briefly in the next section.

We expect from a general, relativistic theory that form and physical content of equations describing the nature of gravitation will be highly dependent on the choice of the local reference system. From the vacuum field equations, one may also extract

$$D_{[a\wedge c]}{}^{c} = 0,$$

which has some similarity to Maxwell's equstions [6]. The time-like part of the field equations

$$R_{ab}t^a t^b = -\Theta_c^{\ c}$$

confirms that Θ is traceless. Rewriting the equation above as

$$D_{c\wedge a}{}^{c}t^a - D^{ab}D_{ab} = 0,$$

the time-like part of the field equation appears in a kind of Maxwellian form.

3. THE STATIC SYSTEM

Using a second system remaining at rest with respect to the center of gravitational attraction, one has to calculate the field strength with the help of (2.1) and the inhomogeneous transformation law

$$G_{mn}{}^{r} = A_m^a A_n^b A_c^r A_{ab}{}^{c} + A_c^r A_{n|m}^c.$$

The transport laws are

$$\Phi_{m;n} = \Phi_{m|n} - G_{nm}{}^{r}\Phi_r, \Phi_{m\wedge n} = \Phi_{m|n} - {}^*A_{nm}{}^{r}\Phi_r,$$
$$G_{mn}{}^{r} = {}^*A_{mn}{}^{r} + E_{mn}{}^{r}, E_{mn}{}^{r} = u_m u_n E^r - u_m E_n u^r.$$

${}^*A_{mn}{}^{r}$ is constructed by analogy to (2.4, first line), but the connection is not flat. It contains the geometric quantities

$$B_m = \{1/\alpha r, 0, 0, 0\}, S_m = \{1/\alpha r, (1/r)\cot\vartheta, 0, 0\} \tag{3.1}$$

endowed with some physical properties. The only pure physical quantity is the negative force of gravitational attraction:

$$E_\alpha = (\ln\alpha)_{|\alpha,}, E_m = \{-\alpha M/r^2, 0, 0, 0\}. \tag{3.2}$$

The Riemann tensor has the components

$$R_{rmn}{}^{p} = 2\left[G_{[m\bullet n\bullet;r]}{}^{p} - G_{[m\bullet n\bullet}{}^{t} G_{r]t}{}^{p}\right] = {}^*R_{rmn}{}^{p} + E_{rmn}{}^{p},$$
$${}^*R_{rmn}{}^{p} = 2\left[{}^*A_{[m\bullet n\bullet\wedge r]}{}^{p} + {}^*A_{[m\bullet n\bullet}{}^{t}\,{}^*A_{r]t}{}^{p}\right],$$
$$E_{rmn}{}^{p} = 2\big[E_{[m\bullet n\bullet\wedge r]}{}^{p} - E_{[rm]}{}^{t} G_{tn}{}^{p} - E_{[r\bullet n\bullet}{}^{t} E_{m]t}{}^{p},$$
$$= 2\left[u_n u_{[m} E^p{}_{\wedge r]} - E_{n\wedge[r} u_{m]} u^p + u_n u_{[r} E_{m]} E^p - u^p u_{[r} E_{m]} E_n\right],$$

and the Ricci tensor

$$R_{mn} = {}^*R_{mn} + E_{mn}, E_{[mn]} = 0, \; E_{[m\wedge n]} = 0,$$

$$E_{mn} = E_{rmn}{}^{r} = E_{n\wedge m} - E_n E_m + u_n u_m[E^r{}_{\wedge r} - E^r E_r].$$

The spacelike part of the field equations then is

$${}^*R_{\alpha\beta} + E_{\alpha\beta} = 0, \quad E_{\alpha\beta} = E_{\beta\wedge\alpha} - E_\beta E_\alpha, \quad \alpha = 1, 2, 3. \tag{3.3}$$

The three-dimensional Ricci tensor may be evaluated with ${}^*A_{\alpha\beta}{}^{\gamma}$ and relations (3.1). $E_{\alpha\beta}$ is similar to $\Theta_{\alpha\beta}$ of Chap. 2. Defining the displacment vector

$$\delta x^m = A_\alpha{}^m \delta x^\alpha = \{0, \delta x^{2^\bullet}, \delta x^{3^\bullet}, \delta x^{4^\bullet}\},$$

one finds, for a timelike geodetic deviation,

$$(\delta x^p)_{;mn} u^m u^n = -E^p{}_r \delta x^r.$$

The fourth part of the field equations exhibits the effects of gravitational selfinteraction:

$$R_{mn} u^m u^n = E^r{}_{\wedge r} - E^r E_r = 0 \tag{3.4}$$

or also

$$div\mathbf{E} = \kappa\varepsilon, \varepsilon = \frac{1}{\kappa} E^r E_r, \tag{3.5}$$

The field energy is conserved with respect to local systems: $\varepsilon^\bullet = 0$. Mixed spacelike and timelike parts of the field equations vanish. There is no transport of field energy in the static Schwarzschild system.

4. CONCLUSIONS

We have been successful in proving that Schwarzschild geometry may be reformulated covariantly. Quantities behaving like tensors under restricted transformations satisfy field equations similar to Maxwell's but enriched with energy-like or stress-like expressions satisfying covariant local conservation laws. Generalised Lorentz transformations considerably change the physical picture of the theory and moreover the content of the energy-like expressions. The conservation laws mentioned above make sense only if restricted (system preserving) transformations are performed. Global coordinate systems do not have any physical meaning. They are used only to perform some mathematical operations.

ACKNOWLEDGEMENTS

I would like to thank Prof. H.-J. Treder (Deutsche Akademie der Wissenschaften, Potsdam) for motivating me to write this paper.

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